Most of the power used to move a
ship is needed to push along the bow
wave that builds up in front of the
ship. Ships are designed to use as little
power as possible.

To ensure that the design of a ship is
energy-efficient, designers test models
before the real ship is built. To
calculate the speed to use when testing
a model the following formula is used.

\[ S_m = \frac{S_r \times \frac{L_m}{2}}{\frac{L_r}{2}} \]

where \( S_m \) is the speed of the model in
metres per second, \( S_r \) is the speed of
the real ship in metres per second, \( L_m \)
is the length of the model in metres,
and \( L_r \) is the length of the real ship in
metres. This formula includes powers
with fractional exponents.

**Investigate & Inquire**

Using the power law for exponents, 9 can be written as \( \left(9^{\frac{1}{2}}\right)^2 \), because

\[ \left(9^{\frac{1}{2}}\right)^2 = 9^{\frac{1}{2} \times 2} = 9^1 \text{ or } 9 \]

1. Copy and complete the following statements by replacing each \( \Box \) with a
natural number. The first statement has been partially completed.

   a) \( 9 = \left(9^{\frac{1}{2}}\right)^2 \)
   but \( 9 = (3)^2 \)
   so \( 9^{\frac{1}{2}} = 3 \)
   and \( 9^2 = \Box \)

   b) \( 25 = \left(25^{\frac{1}{2}}\right)^2 \)
   but \( 25 = (5)^2 \)
   so \( 25^{\frac{1}{2}} = (5) \)
   and \( 25^2 = \Box \)

   c) \( 8 = \left(8^{\frac{1}{3}}\right)^3 \)
   but \( 8 = (2)^3 \)
   so \( 8^{\frac{1}{3}} = (2) \)
   and \( 8^3 = \Box \)

   d) \( 16 = \left(16^{\frac{1}{4}}\right)^4 \)
   but \( 16 = (2)^4 \)
   so \( 16^{\frac{1}{4}} = (2) \)
   and \( 16^4 = \Box \)
2. Evaluate.
   a) $36^2$
   b) $27^3$
   c) $81^{\frac{1}{4}}$
   d) $100^{\frac{1}{2}}$

3. A ship is to be built 100 m long and able to travel at 15 m/s. The model of the ship is 4 m long. At what speed should the model be tested?

In the power law for exponents, $(a^m)^n = a^{mn}$, substituting $m = \frac{1}{n}$ gives

$$
\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n} \times n} = a^1 \text{ or } a
$$

If $a \geq 0$, we can take the $n$th root of both sides of the equation $\left(a^{\frac{1}{n}}\right)^n = a$, which gives $a^{\frac{1}{n}} = \sqrt[n]{a}$.

This result suggests the following definition.

$$
a^{\frac{1}{n}} = \sqrt[n]{a}, \text{ where } n \text{ is a natural number.}
$$

The symbol $\sqrt[n]{x}$ indicates an $n$th root, and $\sqrt[n]{x}$ represents the principal $n$th root of $x$. For example, $64^{\frac{1}{3}} = \sqrt[3]{64}$. The expression $\sqrt[3]{64}$ is read as “the cube root of 64.”

Finding the cube root of a number is the inverse operation of cubing. To find the cube root of 64, find the number whose cube is 64. Since $4^3 = 64$, $\sqrt[3]{64} = 4$.

- If $n$ is an even number, then we must have $a \geq 0$ for the $n$th root to be real. Suppose that $n$ is even and $a$ is negative. For example, if $n = 2$ and $a = -4$, then $(-4)^\frac{1}{2}$ becomes $\sqrt{-4}$. There is no real square root of $-4$.
- If $n$ is an odd number, then $a$ can be any real number. For example, if $n = 3$ and $a = -8$, then $(-8)^\frac{1}{3}$ becomes $\sqrt[3]{-8}$, which is $-2$. In this case, the principal root is negative.

Note how brackets are used with fractional exponents. The expression $\sqrt[4]{-4}$ has no meaning in the real number system, but $-\sqrt{4} = -2$. Similarly, $(-4)^\frac{1}{2}$ becomes $\sqrt{-4}$, which has no meaning in the real number system. But $-4^{\frac{1}{2}}$ becomes $-(4^{\frac{1}{2}}) = -\sqrt{4} = -2$. 

12 MHR • Chapter 1
**Example 1** Exponents in the Form $\frac{1}{n}$

Evaluate.

a) $49^{\frac{1}{2}}$

b) $(-27)^{\frac{1}{3}}$

c) $(-8)^{-\frac{1}{3}}$

**Solution**

a) $49^{\frac{1}{2}} = \sqrt{49} = 7$

b) $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

c) $(-8)^{-\frac{1}{3}} = \frac{1}{(-8)^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{-8}} = -\frac{1}{2}$

The following suggests how to evaluate an expression with a fractional exponent in which the numerator is not 1, such as $4^{\frac{3}{2}}$.

The power law $(a^m)^n = a^{mn}$ is used.

**Method 1**

$4^{\frac{3}{2}} = \left(4^1\right)^{\frac{3}{2}}$

$= (4)^{\frac{3}{2}}$

$= 2^3$

$= 8$

**Method 2**

$4^{\frac{3}{2}} = (4^2)^{\frac{1}{2}}$

$= \sqrt[2]{4^2}$

$= \sqrt[2]{4^3}$

$= \sqrt[2]{64}$

$= 8$

Notice that $\left(4^3\right)$ and $\sqrt[2]{4^3}$ have the same value.

This result suggests the following definition for rational exponents.

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$, where $m$ and $n$ are natural numbers.

If $n$ is an even number, then $a \geq 0$.

If $n$ is an odd number, then $a$ can be any real number.

To calculate $a^{\frac{m}{n}}$

- take the $n$th root of $a$, then raise the result to the $m$th power

$g^2 = (\sqrt[2]{9})^3$

$= 3^3$

$= 27$

or
• raise a to the mth power, then take the nth root

\[ g^2 = \sqrt{g^3} \]

\[ = \sqrt{729} \]

\[ = 27 \]

It is common practice to take the nth root first.

**Example 2** Exponents in the Form \( \frac{m}{n} \)

Evaluate.

a) \((-8)^{\frac{4}{3}}\)

b) \(9^{-2.5}\)

c) \(\left(\frac{25}{4}\right)^{\frac{3}{2}}\)

**Solution 1** Paper-and-Pencil Method

a) \((-8)^{\frac{4}{3}} = \left(\sqrt[3]{-8}\right)^4\)

\[ = (-2)^4 \]

\[ = 16 \]

b) \(9^{-2.5} = 9^{\frac{5}{2}}\)

\[ = \left(\sqrt{9}\right)^5 \]

\[ = \frac{1}{3^5} \]

\[ = \frac{1}{243} \]

c) \(\left(\frac{25}{4}\right)^{\frac{3}{2}} = \frac{1}{\left(\frac{25}{4}\right)^{\frac{3}{2}}}\)

\[ = \frac{1}{\left(\frac{25}{4}\right)^{\frac{3}{2}}} \]

\[ = \frac{1}{5^3} \]

\[ = \frac{5^3}{2^3} \]

\[ = \frac{125}{8} \]

\[ = \frac{8125}{125} \]

**Solution 2** Graphing-Calculator Method

The first answer given by a graphing calculator may be a decimal.

If necessary, convert the decimal to a fraction using the Frac function.

a) \((-8)^{\frac{4}{3}}\)

b) \(9^{-2.5}\)

c) \(\left(\frac{25}{4}\right)^{\frac{3}{2}}\)
Example 3  Evaluating Approximate Roots

Use a calculator to evaluate the following, to the nearest hundredth.

a) \(2^{3.5}\)  
b) \(7^{2/3}\)

Solution

\[
\begin{array}{c|c}
\text{Estimate} & \text{Solution} \\
\hline
2^3 & 8 \\
2^4 & 16 \\
2^{3.5} & 12 \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{Estimate} & \text{Solution} \\
\hline
7^{(2/3)} & 3.66 \\
\end{array}
\]

Key Concepts

- \(a^{\frac{1}{n}} = \sqrt[n]{a}\), where \(n\) is a natural number.
- To evaluate \(a^n\) or \(\sqrt[n]{a}\) in the real number system,
  - if \(n\) is even, then \(a \geq 0\),
  - if \(n\) is odd, then \(a\) can be any real number.
- \(a^{\frac{m}{n}} = (\sqrt[n]{a})^m\), where \(m\) and \(n\) are natural numbers.
- To calculate \(a^n\) using paper and pencil, either take the \(n\)th root of \(a\), then raise the result to the \(m\)th power, or raise \(a\) to the \(m\)th power, then take the \(n\)th root.

Communicate Your Understanding

1. Describe how you would evaluate each of the following using paper and pencil.
   a) \(27^{\frac{1}{3}}\)  
   b) \(-27^{\frac{1}{3}}\)  
   c) \((-27)^{\frac{1}{3}}\)

2. Describe two ways to evaluate \(8^{\frac{2}{3}}\) using paper and pencil.

3. Explain whether it is possible to evaluate each of the following in the real number system.
   a) \(16^{\frac{1}{4}}\)  
   b) \((-16)^{\frac{1}{4}}\)  
   c) \((-16)^{\frac{1}{4}}\)
Practise

A

1. Write in radical form.
   a) \(2^{\frac{3}{2}}\)  b) \(37^2\)  c) \(x^2\)
   d) \(a^5\)  e) \(6^3\)  f) \(6^4\)
   g) \(7^{\frac{1}{2}}\)  h) \(9^{\frac{1}{5}}\)  i) \(x^{\frac{3}{7}}\)
   j) \(b^{\frac{6}{5}}\)  k) \((3x)^{\frac{1}{2}}\)  l) \(3x^2\)

2. Write using exponents.
   a) \(\sqrt{7}\)  b) \(\sqrt{34}\)  c) \(\sqrt[3]{-11}\)
   d) \(\frac{5a^2}{b}\)  e) \(\frac{3}{\sqrt{6}}\)  f) \((\frac{b}{a})^4\)
   g) \(\frac{1}{\sqrt[3]{x}}\)  h) \(\frac{1}{\sqrt[3]{a}}\)  i) \(\frac{1}{\sqrt[3]{x^2}}\)
   j) \(\sqrt[3]{2^3}\)  k) \(\sqrt[3]{3x^5}\)  l) \(\sqrt[4]{5t^3}\)

3. Evaluate.
   a) \(4^{\frac{1}{2}}\)  b) \(125^{\frac{1}{3}}\)  c) \(16^{\frac{1}{4}}\)
   d) \((-32)^{\frac{1}{5}}\)  e) \(25^{\frac{3}{2}}\)  f) \((-27)^{\frac{1}{3}}\)
   g) \(64^{\frac{1}{6}}\)  h) \(0.04^{\frac{1}{2}}\)  i) \(81^{\frac{3}{4}}\)
   j) \(0.001^{\frac{1}{3}}\)  k) \((\frac{4}{9})^{\frac{1}{2}}\)  l) \((-27)^{-\frac{1}{3}}\)

4. Evaluate.
   a) \(8^{\frac{3}{2}}\)  b) \(4^{\frac{3}{4}}\)  c) \(9^{2.5}\)
   d) \(81^{\frac{3}{4}}\)  e) \(16^{\frac{3}{4}}\)  f) \((-32)^{\frac{2}{5}}\)
   g) \((-8)^{-\frac{5}{3}}\)  h) \((-27)^{-\frac{3}{2}}\)  i) \(1^{\frac{5}{3}}\)
   j) \((-1)^{\frac{5}{8}}\)  k) \((\frac{100}{9})^{\frac{3}{2}}\)  l) \((\frac{27}{8})^{-\frac{2}{3}}\)

5. Evaluate in the real number system, if possible.
   a) \((-9)^{\frac{1}{2}}\)  b) \(100,000^{\frac{3}{5}}\)
   c) \((\frac{27}{8})^{-\frac{3}{2}}\)  d) \(3^2 \times 3^{\frac{1}{2}}\)

6. Communication Write an equivalent expression using exponents.
   a) \(\sqrt{x^4}\)  b) \(\sqrt[6]{x^6}\)
   c) \(\sqrt[3]{3x^6}\)  d) \(\sqrt[3]{8x^7}\)
   e) \(\sqrt[3]{81x^3}\)  f) \((x^{\frac{1}{3}}y^{\frac{1}{4}})^3\)
   g) \((a^{\frac{1}{2}}b^{\frac{1}{4}})^{12}\)  h) \(\sqrt[3]{-27x}\)
   i) \((81a^6b^4)^{\frac{1}{4}}\)  j) \((27x^6y^9)^{\frac{1}{3}}\)
   k) \((\sqrt[3]{x^2})(\sqrt[3]{x})\)  l) \((\frac{3}{x}^2)(\frac{4}{x^3})\)
   m) \((\sqrt[3]{x^2})(\sqrt[3]{x^2})\)  n) \((\sqrt[3]{a^2b^4})^2\)
   o) \((\sqrt[3]{a^2b^5})^{\frac{1}{2}}\)

7. Estimate. Then, find an approximation for each, the nearest hundredth.
   a) \(6^{0.4}\)  b) \(3^{2.8}\)
   c) \(4^{1.2}\)  d) \(5^{1.3}\)
   e) \(7^{-\frac{3}{5}}\)  f) \(10^{\frac{3}{7}}\)
Apply, Solve, Communicate

8. Ship building  The design of a new ship calls for the ship to be 300 m long and travel at 12 m/s. To test the design, a model 15 m long is used. Using the formula from the beginning of this section, find the speed at which the model should be tested, to the nearest tenth of a metre per second.

9. Horizon  Because the Earth is curved, it is impossible to see beyond the horizon. The distance, \( d \), to the horizon depends on the observer's height, \( h \), above the ground. The radius of the Earth is \( r \). The formula for the distance to the horizon is

\[
d = (2r h + h^2)^{\frac{1}{2}}
\]

a) Use the diagram to show that the formula is valid.
b) Assume that the radius of the Earth is 6370 km. Find the distance to the horizon, to the nearest kilometre, for an observer in an aircraft 10 km above the Earth; in a spacecraft 200 km above the Earth.

10. Weather  Meteorologists have determined that violent storms, such as tornadoes and hurricanes, can be described using the formula \( D = 9.4t^{\frac{2}{3}} \). In this formula, \( D \) kilometres is the diameter of the storm and \( t \) hours is the time for which the storm lasts. If a typical hurricane lasts for about 18 h, what is its diameter, to the nearest kilometre?

11. Mining  The volume of nickel Canada produces in a year is about 21 000 m\(^3\).
a) If this volume of nickel were made into a single cube, what would be the length of each edge, to the nearest tenth of a metre?
b) How does this volume of nickel compare with the volume of your school gymnasium?
12. **Application** The frequency of any note on a piano is measured in vibrations per second, or hertz (Hz). The frequency of each of the other notes in the octave above middle C is a multiple of the frequency of middle C. The table shows the approximate frequency of middle C. Copy and complete the table by finding the approximate frequencies of the other notes, to the nearest tenth of a hertz.

<table>
<thead>
<tr>
<th>Note</th>
<th>Multiple of C</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1</td>
<td>261.6</td>
</tr>
<tr>
<td>C#</td>
<td>$\frac{3}{2}$</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>$(\frac{3}{2})^2$</td>
<td></td>
</tr>
<tr>
<td>D#</td>
<td>$(\frac{3}{2})^3$</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>$(\frac{3}{2})^4$</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>$(\frac{3}{2})^5$</td>
<td></td>
</tr>
<tr>
<td>F#</td>
<td>$(\frac{3}{2})^6$</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>$(\frac{3}{2})^7$</td>
<td></td>
</tr>
<tr>
<td>G#</td>
<td>$(\frac{3}{2})^8$</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>$(\frac{3}{2})^9$</td>
<td></td>
</tr>
<tr>
<td>A#</td>
<td>$(\frac{3}{2})^{10}$</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>$(\frac{3}{2})^{11}$</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>$(\frac{3}{2})^{12}$</td>
<td></td>
</tr>
</tbody>
</table>

13. **Equations** Evaluate $x$, where $x$ is a natural number.

- a) $2^x = 32$
- b) $3^{x+1} = 81$
- c) $(-1)^x = 1$
- d) $6^{x-2} = 36$
- e) $2^{2x} = 16$
- f) $(-1)^x = -1$

14. **Inquiry/Problem Solving**

- a) The diagrams show 2 squares with whole-number areas that can be made on a 4-pin by 4-pin geoboard. If the shortest distance between 2 pins is 1 unit, what is the area of each square?
- b) Draw the 3 other different-sized squares with whole-number areas that can be made on the same geoboard.
- c) Of the 5 different-sized squares, which ones do not have whole-number side lengths? Express their side lengths using fractional exponents.
- d) Draw the 8 different-sized squares with whole-number areas that can be made on a 5-pin by 5-pin geoboard. For the squares that do not have whole-number side lengths, express the side lengths using fractional exponents.
- e) Repeat part d) for a 6-pin by 6-pin geoboard, and state how many different-sized squares can be made.
- f) Can you make any generalizations or state any conclusions from this investigation?

**Achievement Check**

If $\oplus(a, b, c)$ means $a^b - b^c + c^a$, what does each of the following equal?

- a) $\oplus(1, -1, 2)$
- b) $\oplus\left(\frac{1}{3}, -1, 8\right)$
- c) $\oplus(-0.5, x, 4)$