1.3 Solving Exponential Equations

Radioactive isotopes have many uses, including medical diagnoses and treatments, and the production of nuclear energy.

Different radioactive isotopes decay at different rates. The time it takes for half of any sample of an isotope to decay is called the half-life of the isotope. The decay of radioactive isotopes is described by the following equation.

\[ A_L = A_o \left( \frac{1}{2} \right)^{\frac{t}{H}} \]

where \( A_L \) is the amount of the isotope left, \( A_o \) is the original amount of the isotope, \( t \) is the elapsed time, and \( H \) is the half-life of the isotope.

The half-life of tungsten-187 is 1 day, so the decay rate for tungsten-187 is described by the following equation.

\[ A_L = A_o \left( \frac{1}{2} \right)^t \]

where \( t \) is the elapsed time in days.

The above equations are examples of exponential equations. These are equations in which the variables appear as exponents.

**Investigate & Inquire**

To solve the equation \( 8^{x-2} = 2^{x+4} \) means to determine the value of \( x \) that makes the equation true, or satisfies the equation.

1. Use a table and guess and check to determine the value of \( x \) that satisfies this equation. The first row has been completed for you.

<table>
<thead>
<tr>
<th>Value of x</th>
<th>L.S. ( = 8^{x-2} )</th>
<th>R.S. ( = 2^{x+4} )</th>
<th>Does L.S. = R.S.?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 8^{1-2} = 8^{-1} = \frac{1}{8} )</td>
<td>( 2^{1+4} = 2^5 = 32 )</td>
<td>No, R.S. &gt; L.S.</td>
</tr>
</tbody>
</table>

2. a) On the left side of the equation \( 8^{x-2} = 2^{x+4} \), replace the base 8 with a power of 2.

b) Use an exponent law to simplify the exponent on the left side of the equation.

c) Since the bases on both sides are equal, how must the exponents be related to make the equation true?

d) Write a linear equation using the two exponents, and solve the equation.
3. Is the solution you found in question 2d) the same as the one you found by guess and check?

4. Using your results from questions 2 and 3, write a rule for solving exponential equations.

5. Test your rule by using it to solve each of the following. Use substitution to verify each solution.
   a) $2^{x+1} = 4^{x-1}$
   b) $9^{x+4} = 27^{2x}$
   c) $8^{2x-3} = 16^{1-x}$
   d) $25^x = 5^{3x}$
   e) $2^{4x-1} = 4^x$

6. Suppose an original sample of tungsten-187 had a mass of 64 mg, and there are 2 mg left.
   a) Substitute the given mass values into the equation $A_L = A_0 \left(\frac{1}{2}\right)^t$ described above.
   b) Express $\left(\frac{1}{2}\right)^t$ as a power of 2.
   c) Express the right side of the equation as a single power of 2.
   d) Use your rule from question 4 to solve the equation for $t$. What was the elapsed time?
   e) Check your solution by substitution in the original equation.

One method for solving an exponential equation is to rewrite the powers with the same base, so that the exponents are equal. Equating the exponents gives a linear equation, which can be solved.

This method of solving an exponential equation is based on the property that, if $a^x = a^y$, then $x = y$, for $a \neq 1, 0, -1$.

**Example 1** Solving Using a Common Base

Solve and check $4^{x+1} = 2^{x-1}$.

**Solution**

The base 4 on the left side is a power of 2.

$4^{x+1} = 2^{2x-1}$

Rewrite using base 2: $(2^2)^{x+1} = 2^{x-1}$

Simplify exponents: $2^{2x+2} = 2^{x-1}$

Equate exponents: $2x + 2 = x - 1$

Solve for $x$: $x = -3$
Check.

**L.S.** = \(4^{x + 1}\)  
**R.S.** = \(2^{x - 1}\)

\[
= 4^{-3 + 1} = 2^{-3 - 1} \\
= 4^{-2} = 2^{-4} \\
= \frac{1}{4^2} = \frac{1}{2^4} \\
= \frac{1}{16} = \frac{1}{16}
\]

L.S. = R.S.

The solution is \(x = -3\).

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**Example 2: Rational Solutions**

Solve and check \(9^{3x + 1} = 27^x\).

**Solution**

Both bases are powers of 3.

\[
9^{3x + 1} = 27^x \\
(3^2)^{3x + 1} = (3^3)^x \\
3^{6x + 2} = 3^{3x}
\]

Equate exponents:

\[
6x + 2 = 3x \\
3x = -2 \\
x = -\frac{2}{3}
\]

The solution is \(x = -\frac{2}{3}\).

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**Example 3: Solving Using a Common Factor**

Solve and check \(3^{x + 2} - 3^x = 216\).

**Solution**

Remove a common factor:

\[
3^x(3^2 - 1) = 216 \\
3^x(8) = 216 \\
3^x = 27 \\
x = 3
\]

The solution is \(x = 3\).
EXAMPLE 4 Modelling Exponential Decay

A radioactive isotope, iodine-131, is used to determine whether a person has a thyroid deficiency. The iodine-131 is injected into the blood stream. A healthy thyroid gland absorbs all the iodine. The half-life of iodine-131 is 8.2 days, so its decay can be modelled by the exponential equation

\[ A_L = A_0 \left( \frac{1}{2} \right)^{\frac{t}{8.2}} \]

where \( A_L \) is the amount of iodine-131 left, \( A_0 \) is the original amount of iodine-131, and \( t \) is the elapsed time, in days. After how long should 25% of the iodine-131 remain in the thyroid gland of a healthy person?

**Solution**

25% = \( \frac{1}{4} \)

so \( A_L = \frac{1}{4}A_0 \)

Write the equation:

\[ A_L = A_0 \left( \frac{1}{2} \right)^{\frac{t}{8.2}} \]

Substitute \( \frac{1}{4}A_0 \) for \( A_L \):

\[ \frac{1}{4}A_0 = A_0 \left( \frac{1}{2} \right)^{\frac{t}{8.2}} \]

Divide both sides by \( A_0 \):

\[ \frac{1}{4} = \left( \frac{1}{2} \right)^{\frac{t}{8.2}} \]

Rewrite using base \( \frac{1}{2} \):

\[ \left( \frac{1}{2} \right)^2 = \left( \frac{1}{2} \right)^{\frac{t}{8.2}} \]

Equate exponents:

\[ 2 = \frac{t}{8.2} \]

Solve for \( t \):

\[ 16.4 = t \]

So, 25% of the iodine-131 should remain in the thyroid gland of a healthy person after 16.4 days.

Check.

After one half-life, \( \frac{1}{2} \) or 50% is left.

After two half-lives, \( \frac{1}{4} \) or 25% is left.

Two half-lives is \( 2 \times 8.2 \) days, or 16.4 days.
Communicate Your Understanding

1. Explain why the second key concept, above, includes “a ≠ 1, 0, –1.”
2. Describe how you would solve $2^{x+3} = 4^{x-1}$.
3. Describe how you would solve $2^{x+1} + 2^x = 48$.
4. To solve $3^{-x} = 1$ using a common base, how would you rewrite it?

Practise

A

1. Solve.
   a) $2^x = 16$
   b) $3^y = 27$
   c) $2^x = 128$
   d) $5^y = 125$
   e) $4^y = 256$
   f) $729 = 9^x$
   g) $(-3)^y = -27$
   h) $(-2)^y = -32$
   i) $(-5)^x = 25$
   j) $81 = (-3)^y$
   k) $-2^x = -16$
   l) $-4^y = -64$
   m) $-5^x = -625$
   n) $(-1)^y = 1$
   o) $(-1)^m = -1$

2. Solve.
   a) $7^{x-2} = 49$
   b) $3^{x+4} = 27$
   c) $2^{1-x} = 128$
   d) $4^{3x} = 64$
   e) $5^{3x-1} = 25$
   f) $-81 = -3^{2x+8}$
   g) $4^{x-1} = 1$
   h) $3^{2-2x} = 1$
   i) $(-1)^{2x} = 1$

3. Solve and check.
   a) $6^{x+3} = 6^x$
   b) $2^{x+3} = 2^{2x+1}$
   c) $3^{2x+3} = 3^y+5$
   d) $2^{4x-7} = 2^{2x+1}$
   e) $7^{5d-1} = 7^{2d+5}$
   f) $3^{b-5} = 3^{2b-3}$

4. Solve.
   a) $16^{2x} = 8^{3x}$
   b) $4^i = 8^{i+1}$
   c) $27^{x-1} = 9^{2x}$
   d) $25^{2-c} = 125^{2c-4}$
   e) $16^{2p+1} = 8^{3p+1}$
   f) $(-8)^{1-2x} = (-32)^{1-x}$

5. Solve and check.
   a) $2^{x+5} = 4^{x+2}$
   b) $2^x = 4^{x-1}$
   c) $9^{2a-6} = 3^{2a+6}$
   d) $4^y = 8^{y+1}$
   e) $27^{y-1} = 9^{2y-4}$
   f) $8^{x+3} = 16^{2x+1}$

6. Solve and check.
   a) $5^{4-x} = \frac{1}{5}$
   b) $10^{y-2} = \frac{1}{10000}$
   c) $6^{3x-7} = \frac{1}{6}$
   d) $3^{3x-1} = \frac{1}{81}$
   e) $5^{2n+1} = \frac{1}{125}$
   f) $\frac{1}{256} = 2^{-3w}$

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7. Solve and check.
   a) $4^x = 8$
   b) $64^x = 16$
   c) $(-8)^y = -2$
   d) $9^{-x} = 3$
   e) $2^{9x} = \frac{1}{8}$
   f) $9^{6x} = \frac{1}{27}$
   g) $2^x = 16^4$
   h) $2^{-2g} = 32$
   i) $9^{2x+1} = 27$

8. Solve and check.
   a) $9^{x+1} = 27^{2x}$
   b) $16^y = 64^{2y-1}$
   c) $36^{1-x} = 216^{-2t}$
   d) $8^{2x-1} = 16^{x-1}$
   e) $25^{1-3x} = 125^{-x}$
   f) $16^{x+k} = 32^{1-2k}$

9. Solve and check.
   a) $5 = 25^{\frac{x}{2}}$
   b) $8 = 2^{\frac{x}{3}}$
   c) $9^{\frac{3}{2}} = 27$
   d) $\frac{1}{2} = 2^{\frac{9}{2}}$
   e) $4^x = \frac{1}{8}$
   f) $\left(\frac{3}{2}\right)^m = \frac{4}{9}$

10. Solve.
    a) $3(5^{x+1}) = 15$
    b) $2(3^{x-2}) = 18$
    c) $5(4^x) = 10$
    d) $2(4^{x+1}) = 1$
    e) $2 = 6(3^{4x-2})$
    f) $27(3^{3x+1}) = 3$

11. Solve and check.
    a) $2^{x+2} - 2^x = 48$
    b) $4^{x+3} + 4^x = 260$
    c) $2^{x+5} + 2^{x} = 1056$
    d) $6^{x+1} + 6^{x+2} = 7$
    e) $3^{x+3} - 3^{x+1} = 648$
    f) $10^{x+4} + 10^{x+3} = 11$
    g) $2^{x+2} - 2^{x+5} = -7$
    h) $3^{m+1} + 3^{m+2} - 972 = 0$
    i) $5^{n+2} - 5^{n+3} = -2500$

Apply, Solve, Communicate

12. Communication  Solve $4^{3x+3} = 8^{2x+2}$. Explain your answer.

13. Half-life  The half-life of ruthenium-106 is 1 year, so the decay of ruthenium-106 is described by the exponential equation
    \[ A_L = A_o \left(\frac{1}{2}\right)^t \]
    where \( t \) is the elapsed time, in years. If an original sample of ruthenium-106 had a mass of 128 mg, and there are 2 mg left, what is the elapsed time?

14. Paper industry  Strontium-90 is used in machines that control the thickness of paper during the manufacturing process. Strontium-90 has a half-life of 28 years. Determine how much time has elapsed if the following fraction of a strontium-90 sample remains.
    a) $\frac{1}{4}$
    b) $\frac{1}{8}$
    c) $\frac{1}{32}$
15. Application  The biological half-life of thyroid hormone T4 is about 6.5 days. If a dose of T4 was not followed by repeat doses,
   a) what fraction of the original dose would remain in the body after 19.5 days?
   b) how long would it take until only 6.25% of the original dose would remain in the body?

16. Scuba diving  The percent of sunlight, $s$, that reaches a scuba diver under water can be modelled by the equation
   $s = 0.8^d \times 100$
   where $d$ is the depth of the diver, in metres.
   a) At what depth does 64% of sunlight reach the diver?
   b) What percent of sunlight reaches the diver at a depth of 10 m, to the nearest percent?

17. Application  Determine the half-life of each isotope.
   a) In 30 h, a sample of plutonium-243 decays to $\frac{1}{64}$ of its original amount.
   b) In 40.8 years, a sample of lead-210 decays to 25% of its original amount.
   c) In 2 min, a sample of radium-221 decays to 6.25% of its original amount.

18. Circulation  Sodium-24 is used to diagnose circulatory problems. The half-life of sodium-24 is 14.9 h. A hospital buys a 40-mg sample of sodium-24. After how long will only 2.5 mg remain?

19. Solve.
   a) $\frac{27^x}{9^{2x-1}} = 3^{x+4}$  b) $27^x(9^{2x-1}) = 3^{x+4}$  c) $27^x + 1 = \left(\frac{1}{9}\right)^{2x-5}$

20. Solve.
   a) $x^2 + 2x = 2x + 6$  b) $3x - 2x = 3^{x-2}$  c) $2^{2x-3}x = 2^{x^2 - 2x + 12}$

21. Half-life  In 8 days, a sample of vanadium-48 decays to $\frac{1}{\sqrt{2}}$ of its original amount. Determine the half-life of vanadium-48.

22. Solve and check.
   a) $\frac{2^{x+1}}{2^{x-3}} = 4$  b) $\frac{9^{x+4}}{27^{x-1}} = 81$  c) $8^{x+2} = 4^{x+3}$

23. Find $x$ and $y$ if $\frac{16^{x+2y}}{8^{x-y}} = 32$ and $\frac{32^{x+3y}}{16^{x+2y}} = \frac{1}{8}$.
CAREER CONNECTION  Microbiology

The science of microbiology is the study of micro-organisms. These are organisms that are too small to be examined with the naked eye. They were first studied after the development of microscopes. Examples of micro-organisms include bacteria, fungi, algae, and viruses.

Bacteria are widely feared because some of them cause diseases in living things. However, we make use of many other bacteria in our daily lives. For example, bacteria are used to make yogurt and cheese from milk, to treat sewage, and to make antibiotics, such as penicillin.

1. Bacteria  
   a) The number of bacteria in a culture is doubling every 7 h. Explain how the equation $N = N_0 (2)^{t/7}$ models the number of bacteria in the culture. Define each term in the equation.
   b) In the culture from part a), if there are 100 000 bacteria at a certain time, how many hours later will the number of bacteria be 800 000? 6 400 000? 25 600 000?
   c) In a different culture, the number of bacteria increases from 15 000 to 240 000 in 24 h. How much longer will it take for the number of bacteria to reach 480 000?
   d) Write an equation that models the number of bacteria in the culture in part c).

2. Research  
   Use your research skills to investigate the following.
   a) the education and training needed for a career in microbiology, and the employers who hire microbiologists
   b) an aspect of microbiology that is important in Canada

NUMBER Power

You have 1023 coins. How can you place them in 10 bags so that, if you are asked for any number of coins from 1 to 1023, you can provide the number without opening a bag?