1.8 Adding and Subtracting Rational Expressions, II

The three-toed sloth of South America moves very slowly. It can travel twice as fast in a tree as it can on the ground. If its speed on the ground is $s$ metres per minute, its speed in a tree is $2s$ metres per minute. The sloth can travel 15 m on the ground in \( \frac{15}{s} \) minutes and 15 m in a tree in \( \frac{15}{2s} \) minutes. The total time it takes to travel 15 m on the ground and 15 m in a tree is \( \frac{15}{s} + \frac{15}{2s} \) minutes. The expression \( \frac{15}{s} + \frac{15}{2s} \) is the sum of the two rational expressions with different denominators. Adding these rational expressions involves finding the LCM of two monomials that include variables.

**Investigate & Inquire**

1. To find the LCM of each group of monomials, copy and complete the table.

<table>
<thead>
<tr>
<th>Monomials</th>
<th>Factored Form</th>
<th>LCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2ab) (\text{b}^2)</td>
<td>(2 \times a \times a \times b)</td>
<td>(2 \times 3 \times b \times b)</td>
</tr>
<tr>
<td>(\frac{10x^3}{15x^2y^4})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3xy) (6yz) (9xz)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{5x^3}{8x^2y}) (10xy^2)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Describe a method for mentally finding the LCM of monomials that include variables.

3. a) Factor the binomials \(2x + 4\) and \(3x + 6\).

   b) Write the LCM of the binomials in factored form.
4. Write the LCM of each pair of expressions in factored form.
   a) $3a - 9, 4a - 12$
   b) $x^2 + x, 2x^2 + 2x$
   c) $2y - 6, y^2 - 9$
   d) $x^2 + 5x + 6, x^2 + x - 2$

5. The total time a three-toed sloth takes to travel 15 m on the ground and 15 m in a tree is \( \frac{15}{s} + \frac{15}{2s} \) minutes.
   a) State the common denominator of the two rational expressions.
   b) Add the expressions.
   c) If \( s \) represents 2.5 m/min, what is the total time, in minutes, the sloth takes to travel 15 m on the ground and 15 m in a tree?

**Example 1**  Adding and Subtracting With Monomial Denominators

Simplify \( \frac{4}{5a} - \frac{3}{2a^2} + \frac{1}{a^3} \). State the restriction on the variable.

**Solution**

Find the LCD.

\[
\begin{align*}
5a &= 5 \times a \\
2a^2 &= 2 \times a \times a \\
a^3 &= a \times a \times a \\
\text{The LCD is } 5 \times 2 \times a \times a \times a = 10a^3.
\end{align*}
\]

Write with the common denominator:

\[
\frac{4}{5a} - \frac{3}{2a^2} + \frac{1}{a^3} = \frac{2a^2(4) - 5a(3) + 10(1)}{2a^2(5a)}
\]

\[
= \frac{8a^2 - 15a + 10}{10a^3}
\]

Add or subtract the numerators:

\[
= \frac{8a^2 - 15a + 10}{10a^3}
\]

Exclude the values for which \( 5a = 0 \) or \( 2a^2 = 0 \), or \( a^3 = 0 \).

\[
\begin{align*}
a &= 0 \\
a &= 0 \\
a &= 0
\end{align*}
\]

So, \( a \neq 0 \).

Therefore, \( \frac{4}{5a} - \frac{3}{2a^2} + \frac{1}{a^3} = \frac{8a^2 - 15a + 10}{10a^3}, \ a \neq 0 \).
**Example 2** Denominators With a Common Binomial Factor

Simplify \( \frac{m}{2m - 4} - \frac{3}{3m - 6} + 1 \). State the restriction on the variable.

**Solution**

\[
2m - 4 = 2(m - 2) \\
3m - 6 = 3(m - 2)
\]

The LCD is \( 2 \times 3 \times (m - 2) \) or \( 6(m - 2) \).

\[
\frac{m}{2m - 4} - \frac{3}{3m - 6} + 1 = \frac{m}{2(m - 2)} - \frac{3}{3(m - 2)} + 1
\]

Write with the common denominator:

\[
= \frac{3(m)}{3 \times 2(m - 2)} - \frac{2(3)}{2 \times 3(m - 2)} + \frac{6(m - 2)(1)}{6(m - 2)}
\]

Add or subtract the numerators:

\[
= \frac{3m - 6 + 6m - 12}{6(m - 2)}
\]

Expand the numerator:

\[
= \frac{9m - 18}{6(m - 2)}
\]

Simplify:

\[
= \frac{3 \times 3 \times (m - 2)}{3 \times 2 \times (m - 2)}
\]

Factor:

\[
= \frac{3 \times 3 \times (m - 2)}{3 \times 2 \times (m - 2)}
\]

Divide by the common factors:

\[
= \frac{3}{2}
\]

Exclude the values for which \( 2m - 4 = 0 \) or \( 3m - 6 = 0 \).

\[
m = 2 \\
m = 2
\]

So, \( m \neq 2 \).

Therefore, \( \frac{m}{2m - 4} - \frac{3}{3m - 6} + 1 = \frac{3}{2}, m \neq 2 \).
**Example 3** Denominators With Different Binomial Factors

Simplify \( \frac{x}{6x + 6} + \frac{5}{4x - 12} \). State the restrictions on the variable.

**Solution**

\[ 6x + 6 = 6(x + 1) \]
\[ 4x - 12 = 4(x - 3) \]

The LCD is \( 12(x + 1)(x - 3) \).

\[ = \frac{x}{6(x + 1)} + \frac{5}{4(x - 3)} \]

Write with the common denominator:

\[ = \frac{2(x - 3)}{2(x - 3)} \times \frac{x}{6(x + 1)} + \frac{3(x + 1)}{3(x + 1)} \times \frac{5}{4(x - 3)} \]

\[ = \frac{2x(x - 3) + 15(x + 1)}{12(x + 1)(x - 3)} \]

Add the numerators:

\[ = \frac{2x^2 - 6x + 15x + 15}{12(x + 1)(x - 3)} \]

Expand the numerator:

\[ = \frac{2x^2 + 9x + 15}{12(x + 1)(x - 3)} \]

Simplify:

\[ \text{Exclude the values for which } 6x + 6 = 0 \text{ or } 4x - 12 = 0. \]

\[ x = -1 \quad x = 3 \]

So, \( x \neq -1, 3 \).

Therefore, \( \frac{x}{6x + 6} + \frac{5}{4x - 12} = \frac{2x^2 + 9x + 15}{12(x + 1)(x - 3)} \), \( x \neq -1, 3 \).

**Example 4** Trinomial Denominators

Simplify \( \frac{4}{y^2 + 5y + 6} - \frac{5}{y^2 - y - 12} \). State the restrictions on the variable.

**Solution**

\[ y^2 + 5y + 6 = (y + 2)(y + 3) \]
\[ y^2 - y - 12 = (y + 3)(y - 4) \]

The LCD is \( (y + 2)(y + 3)(y - 4) \).
\[
\frac{4}{y^2 + 5y + 6} - \frac{5}{y^2 - y - 12} = \frac{4}{(y + 2)(y + 3)} - \frac{5}{(y + 3)(y - 4)}
\]

Write with the common denominator:
\[
\frac{y - 4}{y - 4} \times \frac{4}{(y + 2)(y + 3)} - \frac{y + 2}{y + 2} \times \frac{5}{(y + 3)(y - 4)}
\]
\[
= \frac{4(y - 4)}{(y + 2)(y + 3)(y - 4)} - \frac{5(y + 2)}{(y + 2)(y + 3)(y - 4)}
\]

Subtract the numerators:
\[
= \frac{4y - 16 - 5y - 10}{(y + 2)(y + 3)(y - 4)}
\]

Expand the numerator:
\[
= \frac{-y - 26}{(y + 2)(y + 3)(y - 4)}
\]

Simplify:
\[
\frac{4}{y^2 + 5y + 6} - \frac{5}{y^2 - y - 12} = \frac{-y - 26}{(y + 2)(y + 3)(y - 4)}, y \neq -2, -3, 4.
\]

**Key Concepts**

- To add or subtract rational expressions with a common polynomial denominator, write the numerators over the common denominator, and add or subtract the numerators.
- To add or subtract rational expressions with different polynomial denominators, rewrite the expressions with a common denominator. Then, write the numerators over the common denominator, and add or subtract the numerators.

**Communicate Your Understanding**

1. **a)** Describe how you would simplify \( \frac{5}{x^3} + \frac{3}{2x^2} - \frac{7}{3x} \).

   b) What is the restriction on the variable?

2. **a)** Describe how you would simplify \( \frac{5}{x^2 - 4} + \frac{2}{x^2 - x - 2} \).

   b) What are the restrictions on the variable?
Practise

In each of the following, state any restrictions on the variables.

A

1. Write an equivalent expression with a denominator of $12x^2y^2$.
   a) \( \frac{2}{xy} \)
   b) \( \frac{x}{y} \)
   c) \( \frac{5}{3xy^2} \)
   d) \( \frac{-y}{6x^2} \)

2. Find the LCM.
   a) \( 10a^2b, 4ab^3 \)
   b) \( 3m^2n, 2mn^2, 6mn \)
   c) \( 2x^3, 6xy^2, 4y \)
   d) \( 10s^2t, 20s^2t, 15st^2 \)

   a) \( \frac{3}{2x} + \frac{4}{5x} \)
   b) \( \frac{2}{4y} + \frac{3}{3y} - \frac{1}{2y} \)
   c) \( \frac{1}{2x} + \frac{3}{3x} - \frac{2}{x^2} \)
   d) \( \frac{3}{2m^3n} - \frac{1}{m^2n} + \frac{4}{5mn} \)
   e) \( x - \frac{2}{x} + 5 \)
   f) \( \frac{3m + 4}{mn} - \frac{1}{m} - 2 \)
   g) \( \frac{4x - 1}{3x^2} - \frac{2x + 3}{x} + \frac{5x + 2}{5x^2} \)
   h) \( \frac{x - 2y}{x} \bigg/ \frac{4x + y}{xy} - \frac{3x - 4y}{y} \)

4. Find the LCM of each of the following. Leave answers in factored form.
   a) \( 3m + 6, 2m + 4 \)
   b) \( 3y - 3, 5y + 10 \)
   c) \( 4m - 8, 6m - 18 \)
   d) \( 8x - 12, 10x - 15 \)

5. Simplify.
   a) \( \frac{4}{x + 3} + \frac{5}{4x + 12} \)
   b) \( \frac{1}{3y - 15} - \frac{2}{y - 5} \)
   c) \( \frac{t}{t - 4} - \frac{2}{3t - 12} \)
   d) \( \frac{3}{2m + 2} - \frac{5}{3m + 3} \)
   e) \( \frac{2x}{4y - 8} - \frac{3}{3y - 6} \)

   a) \( \frac{2}{x + 1} + \frac{3}{x + 2} \)
   b) \( \frac{m}{m - 3} - \frac{5}{m + 2} \)
   c) \( \frac{3}{x} + \frac{5}{x - 1} \)
   d) \( \frac{2}{t - 1} + \frac{1}{5} + 2 \)
   e) \( \frac{2x}{x - 2} - \frac{3}{x + 2} \)
   f) \( \frac{4}{3n - 1} - \frac{3}{2n + 3} \)
   g) \( \frac{1}{2x - 2} + \frac{3}{4x - 8} \)
   h) \( \frac{t}{3t + 15} - \frac{1}{6t - 24} \)
   i) \( \frac{4}{2s - 12} + \frac{s}{5s - 5} \)
   j) \( \frac{2m}{3m - 15} + \frac{m}{4m - 8} \)

7. State the LCM in factored form.
   a) \( x + 2, x^2 + 4x + 4 \)
   b) \( y^2 + 6y + 8, y^2 - 4 \)
   c) \( x^2 - 12, x^2 - 3x - 4 \)
   d) \( 2x - 4, x^2 - 3x - 4 \)
   e) \( m^2 + 6m + 9, m^2 - 2m - 15 \)

8. Simplify.
   a) \( \frac{2}{x + 3} + \frac{3}{x^2 + 5x + 6} \)
   b) \( \frac{y}{y^2 - 16} - \frac{4}{y + 4} \)
   c) \( \frac{3x}{x - 5} + \frac{2x}{x^2 - 4x - 5} \)
Apply, Solve, Communicate

11. Inquiry/Problem Solving  a) Copy and complete the table. The first line has been completed.

<table>
<thead>
<tr>
<th>Expressions</th>
<th>Product</th>
<th>LCM</th>
<th>GCF</th>
<th>LCM × GCF</th>
</tr>
</thead>
<tbody>
<tr>
<td>3x, 5x</td>
<td>15x²</td>
<td>15x</td>
<td>x</td>
<td>15x²</td>
</tr>
<tr>
<td>12, 8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15y², 9y</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a + 1, a − 1</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>2t − 2, 3t − 3</td>
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</tr>
</tbody>
</table>

b) How is the product LCM × GCF related to the product of each pair of expressions?
c) Explain why the relationship you found in part b) exists.

10. Simplify.

a) \[ \frac{t + 1}{t - 1} + \frac{2}{t^2 - 5t + 4} \]
b) \[ \frac{y + 1}{y - 1} + \frac{y - 1}{y^2 + y - 2} \]
c) \[ \frac{x - 2}{x^2 + 4x + 3} - \frac{2x + 1}{x + 3} \]
d) \[ \frac{n^2 + 4n - 3}{n^2 - 16} + \frac{4 - 3n}{3n - 12} \]
e) \[ \frac{m + 4}{m^2 - m - 12} - \frac{m}{m^2 - 5m + 4} \]
f) \[ \frac{a + 2}{a^2 - 1} - \frac{a - 1}{a^2 + 2a + 1} \]
g) \[ \frac{3w - 4}{w^2 + 5w + 4} + \frac{2w - 3}{w^2 + 2w - 8} \]
h) \[ \frac{2x - 1}{2x^2 + 3x + 1} + \frac{3x^2 - 4x + 1}{3x^2 + 4x + 1} \]
i) \[ \frac{2z - 1}{4z^2 - 25} - \frac{2z + 5}{4z^2 - 8z - 5} \]
12. Application  
a) An RCMP patrol boat left Goderich and travelled for 45 km along the coast of Lake Huron at a speed of \( s \) kilometres per hour. Write an expression that represents the time taken, in hours.

b) The boat returned to Goderich at a speed of 2\( s \) kilometres per hour. Write an expression that represents the time taken, in hours.

c) Write and simplify an expression that represents the total time, in hours, the boat was travelling.

d) If \( s \) represents 10 km/h, for how many hours was the boat travelling?

13. Communication  Write a problem that satisfies the following conditions. Have a classmate solve your problem.
• simplifies using addition and/or subtraction
• includes three rational expressions with different denominators that contain variables
• has the LCD as the denominator of one of the rational expressions


a) \( \frac{m + 3}{m + 2} \times \frac{m + 2}{m + 1} + \frac{5m}{m + 1} \)

b) \( \frac{2x^2 - 2x}{x^2 + 4x - 5} - \frac{4x}{x + 5} \)

c) \( \frac{x^2 + 2x - 15}{x^2 - 7x + 12} + \frac{x^2 - 6x - 7}{x^2 - 3x - 4} \)

d) \( \frac{3y - 1}{y - 4} - \frac{y^2 + 4y - 12}{y^2 - 6y + 8} \)

e) \( \frac{2z^2 - 5z + 3}{z^2 - 1} + \frac{4z^2 - 9}{4z + 6} \)

f) \( \frac{x^2 + 5x + 6}{x^2 - 3x + 2} \div \frac{x + 3}{x - 1} - \frac{6}{x + 3} \)

15. Write two rational expressions with binomial denominators and with each of the following sums. Compare your answers with a classmate’s.

a) \( \frac{5x + 8}{(x + 1)(x + 2)} \)

b) \( \frac{5x - 5}{6x^2 - 13x + 6} \)

c) \( \frac{x^2 - 3}{(x - 1)(x - 3)} \)

d) \( \frac{4x^2}{4x^2 - 9} \)

Achievement Check

Suppose you drive an average of 18 000 km/year. With your present car, you can drive 10 km per litre of fuel. You are thinking of buying a new car that you could drive \( x \) km farther per litre of fuel. Fuel currently costs $0.68/L. If the new car would save you $244.80 in yearly fuel costs, find the number of kilometres you could drive the new car per litre of fuel.