6.6 Geometric Series

The following is a geometric sequence.
3, 6, 12, 24, ..., where $a = 3$ and $r = 2$.

A geometric series is the sum of the terms of a geometric sequence.
The geometric series that corresponds to the geometric sequence above is
$3 + 6 + 12 + 24 + ...$, where $a = 3$ and $r = 2$.
For this series, $S_3$ means the sum of the first 3 terms, so
$S_3 = 3 + 6 + 12$
$= 21$

Investigate & Inquire

Rosa decided to research her ancestry for the last 6 generations, which included her 2 parents, 4 grandparents, 8 great-grandparents, and so on. To go back 6 generations, the total number of people Rosa needed to research was the sum of the following geometric series.

$S_6 = 2 + 4 + 8 + 16 + 32 + 64$

1. a) Use addition to find the total number of people Rosa needed to research.
b) To develop a second method for finding the sum of the series, first write the equation that represents the sum of the first 6 terms and label it equation (1).

\[ S_6 = 2 + 4 + 8 + 16 + 32 + 64 \quad (1) \]

Then, multiply both sides of equation (1) by the common ratio, 2, and label the result equation (2).

\[ 2S_6 = 4 + 8 + 16 + 32 + 64 + 128 \quad (2) \]

Write equations (1) and (2) so that the equal terms on the right side line up as shown.

\[ S_6 = 2 + 4 + 8 + 16 + 32 + 64 \quad (1) \]
\[ 2S_6 = 4 + 8 + 16 + 32 + 64 + 128 \quad (2) \]

Subtract equation (1) from equation (2). Do not simplify the right side.
c) The right side shows the difference between two terms of the series
$2 + 4 + 8 + 16 + ...$ Which two terms?
The method developed in the Investigate & Inquire can be used to write a formula for finding the sum, $S_n$, of the general geometric series.

For the general geometric series,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1},$$

where $S_n$ represents the sum of $n$ terms.

Write the series:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \quad (1)$$

Multiply both sides by $r$:

$$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n \quad (2)$$

Subtract (1) from (2):

$$rS_n - S_n = -a + ar^n$$

Rearrange the right side:

$$rS_n - S_n = ar^n - a$$

Factor the left side:

$$S_n(r - 1) = ar^n - a$$

Divide both sides by $(r - 1)$:

$$S_n = \frac{ar^n - a}{r - 1} \quad r \neq 1$$

The sum is the next term minus the first term divided by the common ratio minus 1.

or $$S_n = \frac{a(r^n - 1)}{r - 1} \quad r \neq 1$$

So, the sum, $S_n$, of the first $n$ terms of a geometric series can be found using the formula

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad r \neq 1$$

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d) Simplify the right side. What is the value of $S_6$?
e) What two numbers would you subtract to find $S_8$ for this series?

2. a) If $S_5 = 1 + 3 + 9 + 27 + 81$, find $S_5$ by addition.
   b) Predict the next term in the series, and subtract the first term from it.
   c) Compare your answer from part b) with your answer from part a).
   d) By what factor would you divide your answer from part b) to give the correct sum?
   e) What is the difference between the factor you found in part d) and the value of the common ratio for this series?

3. Repeat question 2 for the series $S_4 = 3 + 12 + 48 + 192$.

4. Explain why no division step was necessary in question 1 parts d) and e) to find the correct sum.

5. a) Using your results from questions 1 to 4, describe two different methods for finding the sum of a geometric series.
   b) In what situations would it be better to use each method?
**Example 1  Sum of a Geometric Series When \( r > 0 \)**

Find \( S_8 \) for the series \( 2 + 8 + 32 + \ldots \)

**Solution**

\( a = 2, \ r = 4, \) and \( n = 8 \)

\[
S_n = \frac{a(r^n - 1)}{r - 1}
\]

\[
S_8 = \frac{2(4^8 - 1)}{4 - 1}
\]

\[
= \frac{2(4^8 - 1)}{3}
\]

\[
= 43\,690
\]

The sum of the first 8 terms is 43 690.

**Example 2  Sum of a Geometric Series When \( r < 0 \)**

Find \( S_9 \) for the series \( 3 - 9 + 27 \ldots \)

**Solution**

\( a = 3, \ r = -3, \) and \( n = 9 \)

\[
S_n = \frac{a(r^n - 1)}{r - 1}
\]

\[
S_9 = \frac{3((-3)^9 - 1)}{-3 - 1}
\]

\[
= \frac{3((-3)^9 - 1)}{-4}
\]

\[
= 14\,763
\]

The sum of the first 9 terms is 14 763.

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**Example 3  Sum of a Geometric Series Given First and Last Terms**

Find the sum of the series $4 + 12 + 36 + \ldots + 2916$.

**Solution**

To use the formula $S_n = \frac{a(r^n - 1)}{r - 1}$, the number of terms is needed.

For the given series $a = 4$, $r = 3$, and $t_n = 2916$.

Substitute known values: $t_n = ar^{n-1}$

$2916 = 4(3)^{n-1}$

Divide both sides by 4:

$729 = 3^{n-1}$

Write 729 as a power of 3: $3^6 = 3^{n-1}$

Equate the exponents: $6 = n - 1$

Solve for $n$: $7 = n$

The series has 7 terms.

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_7 = \frac{4(3^7 - 1)}{3 - 1}$

$= \frac{4(3^7 - 1)}{2}$

$= 4372$

The sum of the series is 4372.

The sums in Examples 1–3 could be found using a graphing calculator. The following method requires the formula for the $n$th term.

Use the **sequence function** from the **LIST OPS menu** to list the terms to be added. Use the **STO+** key to store the list in L1. Then, use the **sum function** from the **LIST MATH menu** to find the sum.
Key Concepts

• The sum of the first $n$ terms of a geometric series can be found using the formula $S_n = \frac{a(r^n - 1)}{r - 1}$.

• Given at least the first two terms and the last term of a geometric series, the number of terms can be found by substituting known values in $t_n = ar^{n-1}$ and solving for $n$.

Communicate Your Understanding

1. Describe the similarities and the differences between a geometric series and an arithmetic series.

2. Describe how you would find $S_{15}$ for the series $4 + 8 + 16 + ...$

3. Describe how you would find the sum of the series $5 + 15 + 45 + ... + 10935$.

Practise

A

1. Find the indicated sum for each geometric series.
   a) $S_{12}$ for $1 + 2 + 4 + ...$
   b) $S_7$ for $1 + 4 + 16 + ...$
   c) $S_6$ for $3 + 15 + 75 + ...$
   d) $S_8$ for $2 - 6 + 18 - ...$
   e) $S_9$ for $3 - 6 + 12 - ...$
   f) $S_6$ for $256 + 128 + 64 + ...$
   g) $S_7$ for $972 + 324 + 108 + ...$
   h) $S_6$ for $1 - \frac{1}{2} + \frac{1}{4} - ...$

2. Find $S_n$ for each geometric series.
   a) $a = 5, r = 3, n = 8$
   b) $a = 4, r = -3, n = 10$
   c) $a = 625, r = 0.6, n = 5$
   d) $f'(1) = 4, r = 0.5, n = 7$
   e) $a = 100 000, r = -0.1, n = 5$
   f) $f'(1) = \frac{1}{2}, r = -5, n = 6$

3. Find the sum of each geometric series.
   a) $1 + 2 + 4 + ... + 256$
   b) $1 + 3 + 9 + ... + 2187$
   c) $2 - 4 + 8 - ... + 512$
   d) $5 - 15 + 45 - ... + 3645$
   e) $729 + 243 + 81 + ... + 1$
   f) $1200 + 120 + 12 + ... + 0.0012$

4. If $f'(1) = 0.8$ and $f'(2) = 1.6$ for a geometric series, find $S_{10}$.

5. If $f'(1) = 2$ and $f'(2) = -8$ for a geometric series, find $S_{15}$.
6. **Genealogy** Suppose you researched your ancestors back ten generations. How many people would you research?

7. **Pattern** The first four rectangles in a sequence have dimensions of 2 cm by 3 cm, 2 cm by 9 cm, 2 cm by 27 cm, and 2 cm by 81 cm. If the pattern continues, what is the total area of the first 10 rectangles?

8. **Measurement** Larger and larger squares are formed consecutively, as shown in the diagram. The side lengths of the squares form a geometric sequence, starting at 10 cm. For the first 10 squares in the sequence, find
   a) the sum of the perimeters
   b) the sum of the areas

9. **Communication** Some companies use a telephone chain to notify employees that the company is closing because of bad weather. Suppose that, in the first round of calls, the first person in the chain calls four people. Each person called then makes four calls, and so on. What is the total number of people called in the first six rounds of calls?

10. **Lottery** The first prize in a lottery is $100 000. Each succeeding winning number pays 40% as much as the winning number before it. How much is paid out in prizes if 6 numbers are drawn?

11. **Application** The air in a hot-air balloon cools as the balloon rises. If the air is not reheated, the balloon rises more slowly every minute. Suppose that a hot-air balloon rises 50 m in the first minute. In each succeeding minute, the balloon rises 70% as far as it did in the previous minute. How far does the balloon rise in 7 min, to the nearest metre?

12. Find the value of a for each geometric series.
   a) $S_7 = 70 993$ and $r = 4$
   b) $S_6 = -364$ and $r = -3$
   c) $S_5 = 310$ and $r = 0.5$

13. **Billionaire's club** Frank had a plan to become a billionaire. He would put aside 1 cent on the first day, 2 cents on the second day, 4 cents on the third day, and so on, doubling the number of cents each day.
   a) How much money would he have after 20 days?
   b) How many days would it take Frank to become a billionaire?
   c) Do you see any problems with Frank's plan? Explain.
14. **Motion of a pendulum** On the first swing, a pendulum swings through an arc of 40 cm. On each successive swing, the length of the arc is 0.98 of the previous length. In the first 20 swings, what is the total distance that the lower end of the pendulum swings, to the nearest hundredth of a centimetre?

15. **Inquiry/Problem Solving** The sides of the large square are 16 cm. The midpoints of the sides are joined to form a new square. Find the sum of the areas of all the squares.

16. **Bouncing ball** A ball is thrown 16 m into the air. The ball falls, rebounds to half of its previous height, and falls again. If the ball continues to rebound and fall in this manner, find the total distance the ball travels until it hits the ground for the sixth time.

17. A geometric series has three terms. The sum of the three terms is 42. The third term is 3.2 times the sum of the other two. What are the terms?

18. If a sequence is defined by \( f(x) = 4^{x-1} \), where \( x \) is a natural number, find the sum of the first 8 terms.

19. **Algebra** Determine \( S_{15} \) for the series \( 3 + 3x^2 + 3x^4 + \ldots \).

20. **Chess** According to an old tale, the inventor of chess, Sissa Dahir, was granted anything he wished by the Indian king, Shirham. Sissa asked for one grain of wheat for the first square on the chess board, two grains for the second square, four grains for the third, eight grains for the fourth, and so on, for all 64 squares.

   a) How many grains did Sissa ask for?

   b) **Research** If one grain of wheat has a mass of 65 mg, how did the mass of wheat that Sissa asked for compare with the world's annual wheat production?

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**Achievement Check**

In a geometric series, \( t_2 = 36 \) and \( t_6 = 2016 \). Find the sum of the first 8 terms.