For compound interest, the interest is reinvested at regular intervals. The interest is added to the principal to earn interest for the next interval of time, or compounding period. If $4000 is invested at 6.25% per annum, compounded annually, the interest is added to the principal at the end of the year. The next year, interest is earned on the sum of the principal and the interest. Similarly, if $4000 is borrowed at 6.25% per annum, compounded annually, the interest is added to the principal at the end of the year.

Reasons for saving or borrowing are individual. Marc borrows $3000 to take a technology course. Daima compares plans for investing $10,000 in a Registered Retirement Savings Plan. Gabriella invests money so she can upgrade the computers in her business in 5 years.

INVESTIGATE & INQUIRE

Martin and Norma invest $2000 for their granddaughter Linda on her 12th birthday so she will have it on her 18th birthday for her education. The money is in an account earning interest at a rate of 8% per annum, compounded annually.

1. a) Use the formula for simple interest to find how much interest is earned in the first year.
   b) What amount is in the account after 1 year?
   c) Since compound interest is being calculated, the interest from the first year is reinvested with the principal for the next year. How much is the principal for the second year?
   d) Use the formula for simple interest to find the interest on the principal for the second year, to the nearest cent.
   e) What amount is in the account after 2 years?
2. Complete a table like the following for the investment until Linda's 18th birthday. Remember the money is withdrawn on her 18th birthday. Use your answers from question 1 in the second row.

<table>
<thead>
<tr>
<th>Birthday</th>
<th>Year number</th>
<th>Principal ($)</th>
<th>Interest rate</th>
<th>Interest ($)</th>
<th>Amount ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>1</td>
<td>2000.00</td>
<td>0.08</td>
<td>160.00</td>
<td>2160.00</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>2160.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. a) What type of sequence is represented by the values in the Amount column of the table? Explain.
   b) Write the formula for the nth term, $t_n$, for this type of sequence.

4. a) For the sequence representing the amount of Martin and Norma's investment, what is the value of
   i) $a$?  
   ii) $r$?
   b) Use the values of $a$ and $r$ to write a formula for the $n$th term, $t_n$, in this sequence.
   c) To check the formula for the $n$th term, calculate
      i) $t_1$    
      ii) $t_2$     
      iii) $t_3$    
      iv) $t_4$     
      v) $t_7$

5. a) Let $A$ be the amount, $P$ be the principal, $i$ be the interest rate per compounding period, and $n$ be the number of compounding periods. Substitute into the formula for the $n$th term from question 4b to write the formula for compound interest.
   b) Check your formula by substituting values for $P$, $i$, and $n$, and calculating $A$ for this investment.

6. a) If the principal invested is $P$ and the annual interest rate is $i$, what is the amount after each number of years?
   i) 1     
   ii) 2     
   iii) 3     
   iv) 4     
   v) 5
   b) Write the amounts from part a) as a sequence.
   c) What is the value of $a$ for this sequence?
   d) What is the value of $r$?

7. Use your results to describe the relationship between compound interest and the kind of sequence represented by Martin and Norma's investment.
The formula for a geometric sequence can be used to develop the formula for the amount accumulated, $A$, with compound interest,

$$A = P (1 + i)^n.$$

$A$ is the amount at the end of the time for the investment or loan, $P$ is the principal invested, $i$ is the interest rate per compounding period, and $n$ is the number of compounding periods.

**Example 1  Finding the Amount Compounded Annually**

To take a technology course, Marc borrows $3000 at an interest rate of 4.75% per annum, compounded annually. He plans to pay back the loan in 5 years.

a) How much will Marc owe after 5 years?

b) How much interest will Marc pay for the loan?

**Solution 1  Paper-and-Pencil Method**

a) Use the formula for compound interest. Marc's loan is $3000, so $P = 3000$.

The interest rate is 4.75% per annum, compounded annually, so $i = 0.0475$.

The length of time for the loan is 5 years and the interest is compounded annually for 5 compounding periods, so $n = 5$.

$$A = P (1 + i)^n$$

Substitute known values:

$$= 3000(1 + 0.0475)^5$$

Simplify:

$$= 3000(1.0475)^5$$

$$= 3783.48$$

Marc will owe $3783.48 after 5 years.

b) The interest is the amount paid after 5 years less the money borrowed.

$3783.48 - 3000 = 783.48$

Marc will pay $783.48 interest for the loan.

**Solution 2  Graphing-Calculator Method**

a) Change the mode settings to 2 decimal places. From the Finance menu, choose TVM Solver.

Enter the known values.

The investment is for 5 years, so $N = 5$.

The interest rate is 4.75% per annum, so $I = 4.75$. 

Web Connection

www.school.mcgrawhill.ca/resources/

To investigate the power of compounding, visit the above web site. Go to Math Resources, then to MATHEMATICS 11, to find out where to go next. Write a report about simple and compound interest, outlining the similarities and differences.
Marc's loan is $3000, so \( PV = 3000 \).
The interest is compounded annually, so \( C/Y = 1 \).
The interest is calculated at the end of each compounding period, so select \( END \).
Move the cursor to \( FV \) to find the future value of the loan, and press \( \text{ALPHA SOLVE} \). Since Marc will pay out the amount, or future value, \( FV \) is negative.
Marc will owe $3783.48 after 5 years.

b) Marc will pay $783.48 interest for the loan, as shown in part b) of Solution 1.

**Example 2  Comparing the Effect of Different Compounding Periods**

Daima is investing $10000 in a Registered Retirement Savings Plan, or RRSP. She is considering a 9-year plan with an interest rate of 6% per annum, compounded semi-annually, or a 9-year plan with an interest rate of 5.95% per annum, compounded monthly. Which plan should Daima choose? Why?

**Solution 1  Paper-and-Pencil Method**

Use the formula for compound interest.

For 6% per annum, compounded semi-annually, Daima's investment is $10000, so \( P = 10000 \).

The interest is compounded semi-annually, so divide the interest rate per annum by 2 to find the interest rate per compounding period.
\[
0.06 \div 2 = 0.03, \text{ so } i = 0.03.
\]

The investment is for 9 years with interest compounded semi-annually, so multiply the number of years by 2 to find the number of compounding periods.
\[
9 \times 2 = 18, \text{ so } n = 18.
\]

\[
A = P (1 + i)^n
\]

**Substitute known values:**
\[
= 10000(1 + 0.03)^{18}
\]

**Simplify:**
\[
= 10000(1.03)^{18}
\]
\[
= 17024.33
\]

At 6% per annum, compounded semi-annually, Daima's investment would be worth $17024.33 after 9 years.
For 5.95% per annum, compounded monthly, Daima's investment is $10 000, so \( P = 10\ 000 \).

The interest is compounded monthly, so divide the interest rate per annum by 12 to find the interest rate per compounding period.

\[
0.0595 \div 12 \approx 0.004\ 958\ 333, \text{ so } i \approx 0.004\ 958\ 333.\]

The investment is for 9 years with interest compounded monthly, so multiply the number of years by 12 to find the number of compounding periods.

\[
9 \times 12 = 108, \text{ so } n = 108.\]

\[
A = P (1 + i)^n
\]

Substitute known values:

\[
= 10\ 000(1 + 0.004\ 958\ 333)^{108}
\]

Simplify:

\[
= 10\ 000(1.004\ 958\ 333)^{108}
\]

\[
= 17\ 060.43
\]

At 5.95% per annum, compounded monthly, Daima's investment would be $17 060.43 after 9 years.

Daima's investment would be $17 024.33 at 6% per annum, compounded semi-annually, but $17 060.43 at 5.95% per annum, compounded monthly. She should choose the plan with 5.95% per annum, compounded monthly.

**Solution 2  Graphing-Calculator Method**

Change the mode settings to 2 decimal places. From the Finance menu, choose the TVM Solver.

Enter the known values for 6% per annum, compounded semi-annually.

- The investment is for 9 years, so \( N = 9 \).
- The interest rate is 6% per annum, so \( I = 6 \).
- Since the investment is paid out, PV is negative. Daima's investment is $10 000, so \( PV = -10\ 000 \).
- The interest is compounded semi-annually, so \( C/Y = 2 \).
- The interest is calculated at the end of each compounding period, so select END.

Move the cursor to FV to find the future value, and press ALPHA SOLVE.

At 6% per annum, compounded semi-annually, Daima's investment would be worth $17 024.33 after 9 years.
Enter the known values for 5.95% per annum, compounded monthly. The investment is for 9 years, so \( N = 9 \).
The interest rate is 5.95% per annum, so \( I = 5.95 \).
Since the investment is paid out, PV is negative. Daima's investment is $10 000, so \( PV = -10 000 \).
The interest is compounded monthly, so \( C/Y = 12 \).
The interest is calculated at the end of each compounding period, so select END.

Move the cursor to FV to find the future value, and press ALPHA SOLVE.

At 5.95% per annum, compounded monthly, Daima's investment would be $17 060.43 after 9 years.

Since Daima's investment would be $17 024.33 at 6% per annum, compounded semi-annually, and $17 060.43 at 5.95% per annum, compounded monthly, she should choose the plan with 5.95% per annum, compounded monthly.

**Example 3 Finding the Interest Rate Using a Graphing Calculator**

Gabriella hopes the $26 000 she is investing will be worth $40 000 in 5 years to upgrade the computers for her business. What rate of interest, to the nearest hundredth of a per cent, compounded quarterly, would Gabriella need to achieve this goal?

**Solution**

Change the mode settings to 2 decimal places. From the Finance menu, choose TVM Solver.

Enter the known values.
The loan is for 5 years, so \( N = 5 \).
Since the investment is paid out, PV is negative. Gabriella's investment is $26 000, so \( PV = -26 000 \).
The amount wanted in 5 years is $40 000, so \( FV = 40 000 \).
The interest is compounded quarterly, so \( C/Y = 4 \).
The interest is calculated at the end of each compounding period, so select END.

Quarterly means once every quarter of a year, 4 times a year, or every 3 months.
Move the cursor to I to find the interest rate per annum, and press ALPHA SOLVE. Since the number of decimal places is set to 2, the graphing calculator rounds the interest rate to the nearest hundredth of a percent.

Gabriella will need an interest rate of 8.71% per annum, compounded quarterly, to achieve this goal.

**Key Concepts**

- Compound interest is a financial application of geometric sequences.
- The formula for the accumulated amount with compound interest is \( A = P(1 + i)^n \), where \( A \) is the amount at the end of the time for the investment or loan, \( P \) is the principal invested, \( i \) is the interest rate per compounding period, and \( n \) is the number of compounding periods.
- In \( A = P(1 + i)^n \), \( i = r / N \), where \( r \) is the interest rate per annum and \( N \) is the number of compounding periods per year, and \( n = yN \), where \( y \) is the number of years.

**Communicate Your Understanding**

1. Explain the meanings of \( A \), \( P \), \( i \), and \( n \) in the formula for compound interest.
2. Describe how you would find the amount and the interest for a $2000 investment after 5 years at an interest rate of 6% per annum, compounded semi-annually.
3. Suppose $60 000 is borrowed for 3 years at 8% per annum, compounded monthly. Describe how you would find the value of \( i \) and \( n \) to substitute into the formula for compound interest.
4. a) Describe how you would use a graphing calculator to find the interest rate, to the nearest hundredth of a percent, compounded quarterly, that would change the value of a loan from $12 000 to $15 000 in 6 years.
   b) Describe how you would use a graphing calculator to find the length of time, to the nearest month, for $7500 invested at 7% per annum, compounded monthly, to be worth $10 000.
   c) Explain the advantages of using a graphing calculator for parts a) and b).
Practise

A

1. The rate of interest for an investment is 6% per annum. What is the interest rate for each compounding period?
   a) semi-annually  b) quarterly  c) monthly
d) daily in a year that is not a leap year
2. How many compounding periods are there for each loan?
   a) compounding quarterly for 1 year
   b) compounding annually for 1 year
   c) compounding monthly for 3 years
d) compounding annually for 5 years
e) compounding semi-annually for 2 years
f) compounding quarterly for 6 years
3. What is the amount for each loan?
   a) $500 at 5% per annum, compounded annually for 3 years
   b) $45 000 at 10.5% per annum, compounded semi-annually for 5 years
c) $1000 at 4.75% per annum, compounded monthly for 4 years
d) $96 000 at 11% per annum, compounded quarterly for 2 years
e) $140 000 at 9.8% per annum, compounded annually for 7 years
4. Find the amount of the investment and the interest.
   a) $2000 invested for 5 years at 12% per annum, compounded annually
   b) $32 500 invested for 1 year at 8.25% per annum, compounded semi-annually
c) $10 000 invested for 2 years at 5.75% per annum, compounded quarterly
d) $8000 invested for 6 years at 10.5% per annum, compounded monthly
5. Find the amount of each investment.
   a) $2200 for 5 years at 12% per annum, compounded monthly
   b) $4400 for 7 years at 7.25% per annum, compounded annually
c) $12 600 for 4 years at 6.75% per annum, compounded quarterly
d) $500 000 for 10 years at 9.25% per annum, compounded semi-annually

Apply, Solve, Communicate

6. Investing  Reza invested $1000 for a year at 6% per annum. What was the amount if the interest was compounded?
   a) semi-annually?
b) quarterly?
c) monthly?
7. **Borrowing** August borrowed $9500 for 3 years at 11.6% per annum, compounded quarterly.
   a) How much did he owe at the end of 3 years?
   b) How much interest did August pay for the loan?

8. **Summer job** Oscar invests $3200 he won for an essay contest. The investment pays 6.5% per annum, compounded monthly. How much will Oscar have after 18 months?

9. **Stereo** Zaineb is deciding whether to buy a stereo at $695 plus GST and PST now, or to invest the money to buy the stereo in a year. Her account pays 7.35% per annum, compounded monthly. If she can buy the stereo at the same price next year, how much would she save by investing the money?

10. **GIC** Anitha saved $8000 from her first job to buy a Guaranteed Investment Certificate, or GIC, at 5.75%, compounded annually. How much will the GIC be worth after 2 years?

11. **Canada Savings Bond** Marion is saving money for college. She has saved $1585 from her summer job. If she invests in a 2-year plan at her bank, she will earn 4.85% compounded semi-annually. If she buys Canada Savings Bonds, she will earn 6.3%, compounded annually, over 2 years.
   a) Which is the better investment?
   b) What is the difference in the interest?

12. **RRSP** Lila and Paul are investing $5000 in their Registered Retirement Savings Plans, RRSPs, at 6% per annum, compounded quarterly. How much will each investment be worth when they reach 60, if Lila makes the deposit on her 38th birthday and Paul makes the deposit on his 48th birthday?

13. **Vacation** Kate is saving for a vacation in the Bahamas. On January 1, she invested $750 at 8.2% per annum, compounded semi-annually. On July 1, she invested another $750 at the same rate. How much will she have from these investments on the next July 1st?

14. **Travelling** Noha is investing $2517 in an account compounded monthly. She wants to have $3000 in 3 years for a trip to Europe. What interest rate, to the nearest hundredth of a percent, compounded monthly, does she need?
15. **Inquiry/Problem Solving** Research current interest rates and interest rates 20 years ago.
   a) Create a problem about a loan with these rates.
   b) Trade problems with a classmate. Compare solutions.

16. **Geometric sequence** Use the formula for a geometric sequence to solve a question in this section. How does the solution with a geometric sequence compare with your original solution?

17. **Comparing** Simone is comparing an investment of $1550 for 2 years at 5%, compounded annually, 4.95%, compounded semi-annually, and 4.9%, compounded monthly.
   a) Predict which rate would result in the greatest amount and which would result in the least amount.
   b) Check your prediction and order the rates from least to greatest profit.

18. **Application** Joanne is investing $13 600 in an account that pays 8.2% per annum, compounded monthly. A friend has agreed to sell her a second-hand pick-up truck for her landscaping business for $15 900. She will need to pay the government GST on the price of the truck.
   a) How long will it take, to the nearest month, to have enough in the account to buy the truck?
   b) How many months sooner, to the nearest month, could she buy the truck if the interest rate went up to 9.3% per annum, compounded monthly?

19. **Making financial decisions** Suppose your parents invested $4000 on the day you were born, in an account earning interest at a rate of 5.9% per annum, compounded semi-annually. How much would the investment be worth on your 19th birthday? If the rate were compounded monthly instead of semi-annually, what difference would there be to the amount on your 19th birthday?

20. **Communication** a) At what interest rate, to the nearest hundredth of a percent, compounded monthly, would you double an investment of $50 000 in 10 years?
   b) Explain your strategy for part a).
   c) Use an example to illustrate your percent for part a).
21. **Hammurabi’s Code** About 1800 B.C., Hammurabi, King of Babylonia, developed a system of laws with a code that permitted a maximum interest rate of 33% per annum, compounded annually, for loans of grain, and 20% per annum, compounded annually, for loans of silver.

a) To the nearest year, how long would it take for the value of a loan of grain to triple in value?

b) To the nearest year, how long would it take for the value of a loan of silver to triple in value?

22. **True or false** Classify each statement as true or false for investments earning compound interest. Justify your answer with an explanation or with examples.

a) As the compounding period increases, the amount increases.

b) A lower interest rate results in a higher interest.

c) A decrease in the length of time decreases the interest earned.

d) A lower interest rate can result in a higher amount if the number of compounding periods increases.

e) A lower interest rate always results in a higher amount if the number of compounding periods increases.

**Achievement Check**

Paul has deposited money in a savings account paying interest at 5% per annum, compounded semi-annually. If the money was invested at 5% per annum, compounded annually, would it amount to more or less in the same time period? If the money was invested at 5% per annum, compounded quarterly, would it amount to more or less in the same time period? Explain both in general and by example. Find an equivalent rate of interest to 5% per annum, compounded semi-annually if the new rate is compounded annually.