Often, when people invest money, they have a goal for which they want a specific amount of money at a future date. Ramona hopes to buy a house in 3 years and estimates that a down payment of $70 000 should be sufficient. Ravi’s grandparents want to have $150 000 for retirement in 9 years. The Elmview school orchestra plans a fundraising event to make an investment that will provide $8000 toward the cost of a trip to England in 2 years. Olivia compares investment options so she will have $26 000 in 4 years to start her own business.

The principal that is invested or borrowed is called the present value of the investment or loan. The present value that will result in a specific amount, with accumulated interest, can be calculated when the interest rate, the compounding period, and the time that interest is earned or paid are known.

### Investigate & Inquire

Ramona hopes to buy a house in 3 years, and estimates that a down payment of $70 000 should be sufficient. She wants to know how much money to invest now, at 6.25% per annum, compounded annually, to obtain this down payment. The money Ramona invests now is the present value, or PV, of the investment.

The time line shows how the value of the investment increases each year.

<table>
<thead>
<tr>
<th></th>
<th>Now</th>
<th>1 year</th>
<th>2 years</th>
<th>3 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>PV</td>
<td>PV</td>
<td>PV (1 + 0.0625)</td>
<td>PV (1 + 0.0625)²</td>
<td>PV (1 + 0.0625)³</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$70 000</td>
<td></td>
</tr>
</tbody>
</table>

1. a) What is the interest rate per compounding period for Ramona’s investment?

b) What expression on the time line represents the amount of the investment for each of the following?

   i) now    ii) after 1 year    iii) after 2 years    iv) after 3 years
c) What is the amount of the investment after 3 years?

d) Write a formula with the amount from part c) and the equivalent expression from part b).

e) Calculate the present value of Ramona’s investment, to the nearest cent.

2. a) Starting with the value of $PV$ as the first term, list the values of Ramona’s investment in order as a sequence.

b) What type of sequence do these values form? Explain.

c) For the sequence representing Ramona’s investment, what is the value of

i) $a$?

ii) $r$?

d) Use the values of $a$ and $r$ to write the formula for the $n$th term, $t_n$, for the sequence.

e) Use $t_n$ to calculate

i) $t_1$

ii) $t_2$

iii) $t_3$

iv) $t_4$

f) What do you notice about $t_4$?

3. To determine the formula for present value, let $PV$ be the present value or first term, $A$ be the amount at the end of an investment, $i$ be the rate of interest per compounding period, and $n$ be the number of compounding periods. Substitute into the formula for the $n$th term of the sequence for Ramona’s investment. Use a negative exponent to express the formula as a product.

4. a) Write the formula for compound interest.

b) Since the present value, $PV$, of an investment is the principal, $P$, in the compound interest formula, substitute $PV$ for $P$ in the formula for compound interest.

c) Solve the formula for $PV$ and use the exponent rules to write the resulting formula with a negative exponent.

5. Ramona wants to make another investment so that she will have $5000 in 5 years for renovations to the house she will buy. The interest rate for the investment is 7.5% per annum, compounded annually. Find the present value of the amount Ramona wants for renovations using

a) the formula for $t_n$

b) the formula for compound interest, remembering that the principal is the present value

c) the formula for $PV$
6. Use your results to describe the relationship between present value and the kind of sequence represented by Ramona's investment, and to describe the relationship between the formula for present value and the formula for compound interest.

The formula for a geometric sequence can be used to develop the formula for the present value, \( PV \), of an investment or loan with compound interest.

\[
PV = \frac{A}{(1 + i)^n} \quad \text{or} \quad PV = A(1 + i)^{-n}
\]

\( PV \) is the present value, \( A \) is the amount at the end of the investment, \( i \) is the rate of interest per compounding period, and \( n \) is the number of compounding periods.

**Example 1  Finding the Present Value With Interest Compounded Annually**

Ravi's grandparents would like to have $150 000 when they retire in 9 years. How much should they invest now, at an interest rate of 5.75% per annum, compounded annually?

**Solution 1  Paper-and-Pencil Method**

Use the formula for present value.

Ravi's grandparents want $150 000, so \( A = 150 \, 000 \). The interest rate is 5.75% per annum, compounded annually, so \( i = 0.0575 \). The investment is for 9 years compounded annually, so \( n = 9 \).

\[
PV = A(1 + i)^{-n}
\]

Substitute known values: \[= 150 \, 000(1 + 0.0575)^{-9}\]

Simplify: \[= 150 \, 000(1.0575)^{-9}\]
\[= 90 \, 691.77\]

Ravi's grandparents should invest $90 691.77.

**Solution 2  Graphing-Calculator Method**

Change the mode settings to 2 decimal places. From the Finance menu, choose TVM Solver.
Enter the known values.
The investment is for 9 years, so \( N = 9 \).
The interest rate is 5.75% per annum, so \( I = 5.75 \).
Ravi’s grandparents want the amount, or future value, of $150 000, so \( FV = 150 000 \).
The interest is compounded annually, so \( C/Y = 1 \).
The payments are made at the end of each payment interval, so select END.

Move the cursor to PV to find the present value, and press ALPHA SOLVE. Since the investment is paid out, PV is negative.

Ravi’s grandparents should invest $90 691.77.

**Example 2** Finding the Present Value With Interest Compounded Monthly

The Elmview school orchestra is planning a fundraising event to help finance a trip to England in 2 years. The orchestra plans to invest the money from this event for 24 months in an account with an interest rate of 4.5% per annum, compounded monthly. The orchestra hopes the money from this investment will provide $8000 toward the cost of the trip.

a) How much does the orchestra need to raise to achieve this goal?
b) How much interest should the orchestra earn to meet this goal?

**Solution 1** Paper-and-Pencil Method

a) Use the formula for present value.
The orchestra wants $8000, so \( A = 8000 \).
The interest rate is 4.5% per annum, compounded monthly, so divide the interest rate by 12.
\[ 0.045 \div 12 = 0.00375, \text{ so } i = 0.00375. \]
The investment is for 2 years compounded monthly for \( 2 \times 12 = 24 \) compounding periods, so \( n = 24 \).
\[ PV = A(1 + i)^{-n} \]

Substitute known values:
\[ = 8000(1 + 0.00375)^{-24} \]
Simplify:
\[ = 8000(1.00375)^{-24} \]
\[ = 7312.68 \]

The orchestra needs to raise $7312.68 to achieve this goal.
b) The interest is the amount at the end of the investment less the amount invested.
\[ I = A - PV \]
\[ = 8000 - 7312.68 \]
\[ = 687.32 \]
The orchestra should earn $687.32 in interest to achieve this goal.

**Solution 2: Graphing-Calculator Method**

**a)** Change the mode settings to 2 decimal places. From the Finance menu, choose TVM Solver.

Enter the known values.
- The interest rate is 4.5% per annum, so \( I = 4.5 \).
- The orchestra wants the amount or future value of $8000, so \( FV = 8000 \).
- The interest is compounded monthly, so \( C/Y = 12 \).
- The payments are made at the end of each payment interval, so select END.

Move the cursor to PV to find the present value, and press ALPHA SOLVE. Since the investment is paid out, PV is negative.

The orchestra needs to raise $7312.68 to achieve this goal.

**b)** The orchestra should earn $687.32 in interest to achieve this goal, as shown in part b) of solution 1.

**Example 3: Comparing the Effects of Different Interest Rates**

Olivia is making arrangements to start her own business in 4 years, and estimates she will need $26 000. She compares investment plans: 5.1% per annum, compounded semi-annually, or 4.9% per annum, compounded quarterly. How much does Olivia need to invest for the better deal?

**Solution 1: Paper-and-Pencil Method**

Use the formula for present value.

For 5.1% per annum, compounded semi-annually:
- Olivia wants $26 000, so \( A = 26 000 \).
- The interest rate is 5.1% per annum, compounded semi-annually.

\[ 0.051 \div 2 = 0.0255 \text{, so } i = 0.0255. \]
The investment is for 4 years compounded semi-annually. 
4 \times 2 = 8 \text{ compounding periods, so } n = 8.

\[
PV = A(1 + i)^{-n}
\]

**Substitute known values:**

\[
= 26 000(1 + 0.0255)^{-8}
\]

**Simplify:**

\[
= 26 000(1.0255)^{-8}
\]

\[
\approx 21 256.32
\]

For 5.1% per annum, compounded semi-annually, Olivia must invest $21 256.32.

For 4.9% per annum, compounded quarterly:

Olivia wants $26 000, so \( A = 26 000. \)

The interest rate is 4.9% per annum, compounded quarterly.

\[
0.049 \div 4 = 0.01225, \text{ so } i = 0.01225.
\]

The investment is for 4 years compounded quarterly.

\[
4 \times 4 = 16 \text{ compounding periods, so } n = 16.
\]

\[
PV = A(1 + i)^{-n}
\]

**Substitute known values:**

\[
= 26 000(1 + 0.01225)^{-16}
\]

**Simplify:**

\[
= 26 000(1.01225)^{-16}
\]

\[
\approx 21 397.78
\]

For 4.9% per annum, compounded quarterly, Olivia must invest $21 397.78.

Olivia needs to invest $21 256.32 for the better deal.

**Solution 2 Graphing-Calculator Method**

Change the **mode settings** to 2 decimal places. From the Finance menu, choose **TVM Solver**.

Enter the known values for 5.1% per annum, compounded semi-annually.

The investment is for 4 years, so \( N = 4. \)

The interest rate is 5.1% per annum, so \( I = 5.1. \)

Olivia wants the amount or future value of $26 000, so \( FV = 26 000. \)

The interest is compounded semi-annually, so \( C/Y = 2. \)

The payments are made at the end of each payment interval, so select END.

Move the cursor to PV to find the present value, and press **ALPHA SOLVE**. Since the investment is paid out, PV is negative.

For 5.1% per annum, compounded semi-annually, Olivia must invest $21 256.32.
Key Concepts

- Present value is a financial application of geometric sequences.
- The formula for the present value, PV, of an investment or loan is
  \[ PV = \frac{A}{(1 + i)^n} \] or \[ PV = A(1 + i)^{-n} \], where PV is the present value, A is the amount at the end of the investment, i is the rate of interest per compounding period, and n is the number of compounding periods.

Communicate Your Understanding

1. Would the present value of a $2000 investment for 2 years, at an interest rate of 8%, compounded quarterly, be higher or lower than if the same investment were compounded semi-annually? Give reasons for your answer.

2. Describe how you would find the present value to have $1000 in 3 years, at an interest rate of 6% per annum, with the interest compounded
   a) annually  
   b) semi-annually  
   c) quarterly  
   d) monthly

3. Explain why the formulas \[ PV = \frac{A}{(1 + i)^n} \] and \[ PV = A(1 + i)^{-n} \] result in the same present value.
**Practise**

**A**

1. What is the rate of interest per compounding period for each investment?
   a) 4.5% per annum, compounded semi-annually  
   b) 5.1% per annum, compounded quarterly  
   c) 8% per annum, compounded annually  
   d) 9% per annum, compounded monthly  

2. An investment with a rate of interest of 6.25% per annum results in $12 000 in 7 years. What is the present value for each compounding period?
   a) annually  
   b) semi-annually  
   c) quarterly  
   d) monthly

3. Consider the present values from question 2. State the relationship between the length of the compounding period and the present value of an investment.

4. What is the present value for each amount?
   a) $9000 in 5 years, invested at 5.6% per annum, compounded semi-annually  
   b) $50 000 in 9 months, invested at 11% per annum, compounded quarterly  
   c) $100 000 in 3 years, invested at 3% per annum, compounded monthly  
   d) $78 840 in 9 years, invested at 4.8% per annum, compounded annually  
   e) $250 000 in a year, invested at 8.75% per annum, compounded quarterly

**Apply, Solve, Communicate**

5. To have $22 000 in 5 years, how much money must be invested today at 5.1% per annum, compounded semi-annually?

6. **Buying a car** How much money should Jessica put into an account paying 8% per annum, compounded semi-annually, to have $17 900 to buy a car in 2 years?

7. **Education** Sue wants to provide for her niece's education. How much should she invest on the day her niece is born to have $22 000 on her 18th birthday, if the money earns 7% per annum, compounded quarterly?

8. **Down payment** Samantha wants to have $40 000 available for a down payment on a house in 10 years. How much should she invest now at 6.25% per annum, compounded semi-annually?

9. **Communication** How much money should Gillian invest now to have $32 000 in 5 years if the money is invested at 8.25% per annum, compounded semi-annually?
10. **Paying off a loan** To pay his tuition, Nathan borrowed money at 3% per annum, compounded semi-annually. For this debt, he owes $5000 to be paid 2 years from now. He earned more at his summer job than he expected, so he wants to pay off the loan at its present value. How much would he pay?

11. **A better deal** Stephanie is choosing an investment plan that will pay $10 000 in 8 years. Does she need to invest more at 6.3% per annum, compounded quarterly, or at 6.3% per annum, compounded monthly? How much more does she need to invest?

12. **Application** An investment rate of 7.4% per annum, compounded annually, is advertised at a bank.
   a) Predict an interest rate compounded quarterly that would result in a present value that is close to the present value for the advertised rate.
   b) Use a graphing calculator to check your prediction with some examples.

13. **Inquiry/Problem Solving** Do you think banks prefer to advertise interest rates for loans that are compounded annually, semi-annually, monthly, or daily? Use the formula for present value, or examples of calculating the present value of investments, to support your answer.

14. a) Explain how the formula for present value is related to the formula for compound interest.
   b) Describe how to use the formula for compound interest to check a solution for finding present value.
   c) Follow your description from part b) to check a solution for a question in this section.
   d) Explain why the formula for a geometric sequence can be used to solve a problem involving present value for compound interest.

15. **Buying a van** A mini-van sells for $32 000 plus GST and PST. A dealership predicts that in 3 years, the cost of the new model will increase by 15%. How much should you invest today at 7.25% per annum, compounded semi-annually, to buy the new model in 3 years?

16. **Changing the rate** Liam’s goal is to save $20 000. What principal invested for 5 years at 6% per annum, compounded semi-annually, then for the next 3 years at 6.5% per annum, compounded quarterly, achieves this goal in 8 years?
17. **Doubling** Consider the formula for present value.
   a) Predict the effect on the present value of an investment of doubling the amount wanted at the end of an investment.
   b) Explain how the formula for present value justifies your prediction for part a).
   c) Demonstrate your prediction with a few examples, using different compounding periods.
   d) Does doubling the length of time for an investment affect the present value in the same way as doubling the amount wanted at the end of the investment? Justify your answer with an explanation or with a few examples.

18. **Re-investing** Marta is investing $6800 at an interest rate of 7% per annum, compounded quarterly, for 2 years. Then, she will invest the amount plus additional money at 6.5% per annum, compounded semi-annually, for 3 years. At the end of the second investment, she wants to have $15 000. How much extra must she invest for the second investment?

19. **Formulating problems** Research two interest rates that might be earned on an investment.
   a) Use the rates to write a problem asking for the present value of $38 000.
   b) Solve your problem.
   c) Trade questions with a few classmates. Compare solutions.

---

### PATTERN Power

The difference $10^2 - 10^1$ equals 90, when expressed in standard form.

1. Express each of the following differences in standard form.
   a) $10^3 - 10^1$  
   b) $10^4 - 10^1$  
   c) $10^3 - 10^2$
   d) $10^4 - 10^3$  
   e) $10^5 - 10^2$  
   f) $10^6 - 10^4$

2. For the difference $10^m - 10^n$ in standard form, where $m$ and $n$ are positive integers and $m > n$,
   a) how many 9s are there?  
   b) how many 0s are there?

3. Use the pattern you found in question 2 to write each of the following differences in standard form.
   a) $10^9 - 10^6$  
   b) $10^{11} - 10^{10}$  
   c) $10^{16} - 10^{12}$

4. Find and describe the pattern for the difference $10^a - 10^b$ in standard form, where $a$ and $b$ are negative integers and $a > b$.

5. Use the pattern you found in question 4 to write each of the following differences in standard form.
   a) $10^{-5} - 10^{-6}$  
   b) $10^{-3} - 10^{-8}$  
   c) $10^{-1} - 10^{-7}$  
   d) $10^{-9} - 10^{-14}$