7.5 Amount of an Ordinary Annuity

Nigel is saving $700 each year for a trip. Rashid is saving $200 at the end of each month for university. Jeanine is depositing $875 at the end of each 3 months for 3 years. Marcel is saving for a home entertainment centre with equal payments at the end of every month for 18 months. These investments by Nigel, Rashid, Jeanine, and Marcel are annuities. An annuity is a series of equal payments at regular intervals of time. For an ordinary annuity, each payment is made at the end of each payment period, or payment interval. A payment interval is the time between successive payments. The word annuity implies annual or yearly payments, however, payment intervals may be any length of time.

Investigate & Inquire

Last June 30, Nigel decided to save for a trip when he graduates. Starting next June 30, and for each of the following 3 years, he plans to deposit $700 into an account that pays 4.5% per annum, compounded annually. Complete the following to find the amount that Nigel will have when he makes the last deposit into this annuity.

The time line shows the value of each deposit at the time of Nigel’s last deposit.

<table>
<thead>
<tr>
<th>Now</th>
<th>June 30</th>
<th>June 30</th>
<th>June 30</th>
<th>June 30</th>
<th>June 30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>700</td>
<td>700(1 + 0.045)^2</td>
<td>700(1 + 0.045)^3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. a) What is the interest rate per period?
b) How many years will the fourth deposit have been in the account?
What will its value be at the time of the last deposit?
c) How many years will the third deposit have been in the account? What will its value be at the time of the last deposit?

d) How many years will the second deposit have been in the account? To the nearest tenth of a cent, what will its value be at the time of the last deposit?

e) How many years will the first deposit have been in the account? To the nearest tenth of a cent, what will its value be at the time of the last deposit?

f) When the last deposit is made, what is the sum of the deposits in Nigel’s account, to the nearest cent?

2. a) What expression on the time line shows the value of each of the following deposits at the time of the last deposit?
   i) the fourth deposit
   ii) the third deposit
   iii) the second deposit
   iv) the first deposit

   b) Start with the expression for the fourth deposit. Write the expressions from part a) in order as the terms of a series.

   c) What type of series is the sum in part b)? Explain.

   d) What is the first term of the series?

   e) What is the common ratio of the series?

   f) How many terms are in the series?

3. a) For the series representing Nigel’s account, what is the value of
   i) \( a \)?
   ii) \( r \)?
   iii) \( n \)?

   b) Use the values of \( a \), \( r \), and \( n \) to write the formula for the sum of the series.

   c) Use the formula to find the amount in Nigel’s account at the time of the last deposit.

4. a) Nigel’s investment is an ordinary annuity. To determine the formula for the amount of an ordinary annuity, let \( A \) be the total amount or sum of the series, \( R \) be the deposit, or payment, made at the end of each compounding period, \( i \) be the interest rate for each compounding period, and \( n \) be the number of compounding periods. Write the formula for the sum, \( A \), of an ordinary annuity by substituting into the formula for the sum of the series for Nigel’s investment. Simplify the denominator.

   b) Use your formula from part a) to find the amount in Nigel’s account at the time of the last deposit.

5. Use your results to describe the relationship between an ordinary annuity and the kind of series represented by the deposits into Nigel’s account.
6. An ordinary annuity consists of a payment of $1000 made on July 20th in 6 successive years, with an interest rate of 7%, compounded annually. Use your formula from question 4 to find the amount of the ordinary annuity on the date of the last payment.

The formula for the sum of a geometric series can be used to develop the formula for the amount, $A$, of an ordinary annuity.

$$A = \frac{R[(1 + i)^n - 1]}{i}$$

$A$ is the amount at the time of the last investment, $R$ is the payment made at the end of each compounding period, $n$ is the number of compounding periods, and $i$ is the interest rate per compounding period.

**Example 1** Finding the Amount of an Annuity Compounded Annually

Starting in 4 months, Jeanine plans to deposit $875 on each July 31, October 31, January 31, and April 30, for 3 years, into an account. With an interest rate of 6%, compounded quarterly, how much will Jeanine have in her account when the last payment is made?

**Solution 1** Paper-and-Pencil Method

Use the formula for an ordinary annuity.

Each payment is $875, so $R = 875$.

There are 4 payments per year for 3 years, so $n = 12$.

The interest rate is 6% per annum, compounded quarterly $0.06 \div 4 = 0.015$, so $i = 0.015$.

$$A = \frac{R[(1 + i)^n - 1]}{i}$$

Substitute known values:

$$= \frac{875[(1 + 0.015)^{12} - 1]}{0.015}$$

Simplify:

$$= \frac{875[(1.015)^{12} - 1]}{0.015}$$

$$= \frac{875[1.1418 - 1]}{0.015}$$

$$= \frac{875 	imes 0.1418}{0.015}$$

$$= 11411.06$$

Jeanine will have $11,411.06 in her account when the last payment is made.
Solution 2  Graphing-Calculator Method

Change the mode settings to 2 decimal places. From the Finance menu, choose TVM Solver.

Enter the known values.
There are 4 payments a year for 3 years, so \( N = 12 \).
The interest rate is 6\% per annum, so \( I = 6 \).
Since the deposits are paid out, PMT is negative. Each deposit is $875, so PMT = −875.
There are 4 payments per year, so \( P/Y = 4 \).
The interest is compounded quarterly, so \( C/Y = 4 \).
The payments are made at the end of each payment interval, so select END.

Move the cursor to FV to find the future value, and press ALPHA SOLVE.

Jeanine will have $11,411.06 in her account when the last payment is made.

Example 2  Finding the Monthly Payment of an Annuity

Marcel wants to buy a home entertainment centre that he sees priced at $3799 plus GST and PST. He plans to buy the centre in 18 months, and he assumes that the price will stay the same. He will make a payment into an account at the end of every month for 18 months. The interest rate is 9\% per annum, compounded monthly.

a) How much will each of Marcel’s payments be?
b) How much interest will Marcel have earned?

Solution 1  Paper-and-Pencil Method

a) Use the formula for an ordinary annuity.
With 7\% GST and 8\% PST, the cost of the home entertainment centre is 115\% of $3799.
\[ 1.15 \times 3799 = 4368.85, \text{ so } A = 4368.85. \]
There is a payment every month for 18 months, so \( n = 18 \).
The interest rate is 9\% per annum, compounded monthly.
\[ 0.09 \div 12, \text{ so } i = 0.0075. \]
\[ A = \frac{R[(1 + i)^n - 1]}{i} \]

Substitute known values: \[ 4368.85 = \frac{R[(1 + 0.0075)^{18} - 1]}{0.0075} \]

Simplify: \[ 4368.85 = \frac{R[(1.0075)^{18} - 1]}{0.0075} \]

Solve for \( R \):

\[ R = \frac{4368.85(0.0075)}{1.0075^{18} - 1} \]

Simplify:

\[ \approx 227.61 \]

Each of Marcel's payments will be $227.61.

b) Marcel will make 18 payments of $227.61.

\[ 18 \times 227.61 = 4096.98 \]

Marcel's payments total $4096.98.

The interest is $4368.85 less $4096.98.

\[ 4368.85 - 4096.98 = 271.87 \]

Marcel will have earned $271.87 in interest.

**Solution 2  Graphing-Calculator Method**

**a)** Change the **mode settings** to 2 decimal places. From the **Finance menu**, choose **TVM Solver**.

Enter the known values.

There is a payment every month for 18 months, so \( N = 18 \).

The interest rate is 9% per annum, so \( I = 9 \).

With 7% GST and 8% PST, the amount, or future value, is 115% of $3799, so \( FV = 1.15 \times 3799 \).

There are 12 payments per year, so \( P/Y = 12 \).

The interest is compounded monthly, so \( C/Y = 12 \).

The payments are made at the end of each payment interval, so select END.

Move the cursor to PMT to find the payment, and press **ALPHA SOLVE**. Since the payment is paid out, PMT is negative.

Each of Marcel's payments will be $227.61.

b) Marcel will have earned $271.87 in interest, as shown in part b) of Solution 1.
**Key Concepts**

- An annuity is a sum of money paid as a series of regular payments. An ordinary annuity is an annuity for which each payment is made at the end of each compounding period.
- The amount of an annuity is a financial application of the sum of a geometric series.
- The formula for the amount, $A$, of an ordinary annuity is 
  
  \[ A = \frac{R[(1 + i)^n - 1]}{i}, \]

  where $A$ is the amount at the time of the last payment, $R$ is the payment made at the end of each compounding period, $n$ is the number of compounding periods, and $i$ is the interest rate per compounding period.

**Communicate Your Understanding**

1. At the end of each month for 2 years, $200 is deposited into an account with an interest rate of 6% per annum, compounded monthly. Describe how you would find the amount of money at the time of the last payment.

2. Starting in 6 months, $1000 is deposited twice a year into an account with an interest rate of 8% per annum, compounded semi-annually. Describe how you would find the amount in 4 years.

3. Explain how the relationship between an ordinary annuity and a geometric series is shown by their formulas.

4. Is the value for the time-value-of-money variable `PMT` on a graphing calculator positive or negative when you are calculating a deposit into an account? Explain.

**Practise**

A

1. Find the number of payments for each investment.
   a) a deposit at the end of every year for 8 years
   b) a deposit at the end of every month for 2 years
   c) a deposit at the end of every 3 months for 15 months

2. For each interest rate, what value would you substitute for $i$ in the annuity formula?
   a) 4% per annum, compounded semi-annually
   b) 3.25% per annum, compounded annually
   c) 9% per annum, compounded quarterly
   d) 6% per annum, compounded monthly
3. For each annuity, what value would you enter for P/Y using the TVM Solver on a graphing calculator?
   a) a deposit at the end of each year for 3 years
   b) a deposit at the end of each month for 5 years
   c) a deposit at the end of each 3 months for 2 years

4. Find the amount of each investment.
   a) $1500 at the end of each year for 6 years, at 7.1% per annum, compounded annually
   b) $300 at the end of each 6 months, for 12 years at 4.95% per annum, compounded semi-annually
   c) 36 monthly payments of $100 at the end of each month, for 3 years at 6% per annum, compounded monthly

5. Find the payment for each ordinary annuity.
   a) 20 semi-annual payments giving an amount of $10 000 at 6% per annum, compounded semi-annually
   b) an amount of $7000 with payments every 3 months for 5 years at 6.15% per annum, compounded quarterly
   c) 36 monthly payments giving an amount of $4000 at 7% per annum, compounded monthly

Apply, Solve, Communicate

6. Saving Starting in 6 months, Lily will deposit $1000 into an account every March 1st and September 1st for 10 years. How much will she have at the time of the last payment if interest is 5.5% per annum, compounded semi-annually?

7. Bank account Marianna deposited $200 into her bank account at the end of each month for 8 months.
   a) The account pays 2.9% per annum, compounded monthly. How much is in her account at the end of the 8 months?
   b) If the amount deposited each month were doubled, how much would be in the account at the end of the 8 months?

8. Pensions David is planning to start saving for his pension by making the same deposit every 6 months starting 6 months after his 35th birthday. The plan he has chosen earns 9% per annum, compounded semi-annually. Use a graphing calculator to find how much each deposit must be to give him half a million dollars on his 60th birthday.

9. Savings account Tian opened a savings account on January 1st with a deposit of $150. The following July 1st, January 1st, and July 1st, he made 3 more deposits of $150 each. The account paid interest at 3.75% per annum, compounded semi-annually.
   a) What amount did Tian have in his account at the end of the second year?
   b) How much interest did Tian earn by the end of the second year?
10. **Application** Shannon plans to buy a new tractor in 3 years. Based on current prices, she predicts a new tractor, including taxes, will cost $90,000 in 3 years. How much should she invest at the end of each month at 9% per annum, compounded monthly, to have enough money to buy the tractor in 3 years?

11. **Communication** Describe a financial situation for paying an annuity as an investment. Research reasonable interest rates.
   a) Write a problem with your information.
   b) Solve the problem from part a).
   c) Trade problems with a classmate. Compare solutions.

12. **Making financial decisions** At the end of grade 9, Rashid set up an annuity to save for university. At the end of each month, he invests $200 into an account bearing interest at 6.25% per annum, compounded monthly. How much money will he have at the end of grade 12?

13. **Inquiry/Problem Solving** Describe a strategy for deciding whether the result you calculate for an annuity is reasonable. Test your strategy.

14. **Comparing solutions** Pierre deposited $200 into his savings account at the end of each month for a year and a half. The account pays 3% per annum, compounded monthly.
   a) Use the formula for the amount of an ordinary annuity to find how much Pierre has in his account at the end of 14 months.
   b) Give the values of \( a \), \( r \), and \( n \) for solving the problem using the formula for the sum of a geometric series. Then, solve the problem with this formula.
   c) Explain how your solutions in parts a) and b) are the same.

15. **Saving for a car** Nelida is purchasing a car for $30,000, including taxes. She hopes to replace it in 4 years with a similar car. She estimates that in 4 years, the price will have increased by 25%, and her present car will have lost 60% of its value. GST of 7% is charged on the difference between the trade-in value and the new car price. PST is charged on the price of the new car. She will start saving in 3 months, by making a payment every 3 months into an account paying 8% interest per annum, compounded quarterly.
   a) How much should each payment be so that she can pay cash for the new car in 4 years?
   b) Explain assumptions you made when finding the payment, and give your opinion about the importance of the assumptions.