7.6 Present Value of an Ordinary Annuity

A lottery advertising a $240,000 prize, in the form of $1000 every month for 20 years, might not have $240,000 when the winning ticket is drawn. Money could be invested in an ordinary annuity to pay $1000 every month for 20 years. The annuity would be earning interest that would be used for the prize of $1000 every month.

The amount of money invested now for an ordinary annuity is the present value of the annuity. The annuity provides equal payments at the end of each equal time interval. The present value can be calculated for a known interest rate, compounding period, number of payments, and amount of each payment.

Investigate & Inquire

Next year, Jane is going back to university for a Ph.D. in psychology. She wants to know how much money to deposit now into an account that pays 6% per annum, compounded annually, to provide a $5000 payment each year for 4 years, with the first payment due a year from now.

The timeline shows the present value of each payment at the time of Jane’s deposit.

<table>
<thead>
<tr>
<th>Now</th>
<th>1 year</th>
<th>2 years</th>
<th>3 years</th>
<th>4 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5000(1 + 0.06)^{-1}$</td>
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<tr>
<td>$5000(1 + 0.06)^{-2}$</td>
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<tr>
<td>$5000(1 + 0.06)^{-3}$</td>
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<tr>
<td>$5000(1 + 0.06)^{-4}$</td>
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</tbody>
</table>

1. a) What is the interest rate per period?
b) How many years will the first payment have been in the account? To the nearest tenth of a cent, how much must be deposited now for this payment?
c) How many years will the second payment have been in the account? To the nearest tenth of a cent, how much must be deposited now for this payment?
d) How many years will the third payment have been in the account? To the nearest tenth of a cent, how much must be deposited now for this payment?
e) How many years will the fourth payment have been in the account? To the nearest tenth of a cent, how much must be deposited now for this payment?
f) To the nearest cent, what is the sum of the amounts that must be deposited now for these payments?

2. a) Which expression on the time line shows the value of each of the following payments at the time of Jane's deposit?
   i) the first payment          ii) the second payment
   iii) the third payment       iv) the fourth payment

b) Start with the expression for the first payment. Write the expressions from part a) in order as the terms of a series.
c) What type of series is the sum in part b)? Explain.
d) What is the first term of the series?
e) What is the common ratio of the series?
f) How many terms are in the series?

3. a) For the series representing Jane's account, what is the value of
   i) a?           ii) r?           iii) n?

b) Use the values of a, r, and n to write the formula for the sum of the series.
c) Use the formula to find the value of Jane's deposit.

4. a) Jane's investment is an ordinary annuity. To determine the formula for the present value of an ordinary annuity, let PV be the present value, R be the payment made at the end of each compounding period, i be the interest rate for each compounding period, and n be the number of compounding periods. Write the formula for the present value, PV, of an ordinary annuity by substituting into the formula for the sum of the series for Jane's investment. Use a negative exponent to express \( \frac{1}{1 + i} \) as \( (1 + i)^{-1} \).

b) Use your formula from part a) to find the amount Jane must invest.

5. Use your results to describe the relationship between an ordinary annuity and the kind of series represented by the payments Jane plans to receive.

6. An ordinary annuity is invested at 4% per annum, compounded annually. The annuity is to pay $1000 a year for 3 years, starting in a year. Use your formula from question 4 to find the present value.
The formula for the sum of a geometric series can be used to develop the formula for the present value, \( PV \), of an ordinary annuity.

\[
PV = R \left[ \frac{1 - (1 + i)^{-n}}{i} \right]
\]

PV is the present value, \( R \) is the payment made at the end of each compounding period, \( n \) is the number of compounding periods, and \( i \) is the interest rate per compounding period.

**Example 1  Finding the Present Value Compounded Semi-Annually**

Michael wants to make an investment so that he would receive $4000 every 6 months for 5 years, with the first payment due in 6 months. How much money should he invest now at 7% per annum, compounded semi-annually?

**Solution 1  Paper-and-Pencil Method**

Use the formula for the present value of an ordinary annuity.

Each payment is $4000, so \( R = 4000 \).
There are 2 payments a year for 5 years, so \( n = 10 \).
The interest rate is 7% per annum, compounded semi-annually.
\( 0.07 \div 2 = 0.035 \), so \( i = 0.035 \).

\[
PV = R \left[ \frac{1 - (1 + i)^{-n}}{i} \right]
\]

Substitute known values:

\[
= 4000 \left[ \frac{1 - (1 + 0.035)^{-10}}{0.035} \right]
\]

Simplify:

\[
= 4000 \left[ \frac{1 - (1.035)^{-10}}{0.035} \right]
\]

\[
\approx 33\,266.42
\]

Michael should invest $33,266.42 now.
SOLUTION 2  Graphing-Calculator Method

Change the mode settings to 2 decimal places. From the Finance menu, choose TVM Solver.

Enter the known values.
There are 2 payments a year for 5 years, so \( N = 10 \).
The interest rate is 7% per annum, so \( I = 7 \).
Michael will receive payments of $4000, so \( PMT = 4000 \).
There are 2 payments per year, so \( P/Y = 2 \).
The interest is compounded semi-annually, so \( C/Y = 2 \).
The payments are made at the end of each payment interval, so select END.

Move the cursor to PV to find the present value, and press ALPHA SOLVE. The investment is paid out, so PV is negative.
Michael should invest $33 266.42 now.

EXAMPLE 2  Finding the Present Value Compounded Monthly

At the end of the season, the jeep Carmen is buying is offered with 0% financing for 48 months. The negotiated cost is $38 400, plus GST and PST. The total cost is divided into equal payments for 48 months, with the first payment on the date of purchase. Carmen will make the first payment, then invest an amount to provide the money each month for the remaining payments, which start in a month. How much must Carmen invest, at 6% per annum, compounded monthly, to have the amount each month for the payment?

SOLUTION 1  Paper-and-Pencil Method

Use the formula for the present value of an ordinary annuity.

With 7% GST and 8% PST, the total cost is 115% of $38 400.
\[ 1.15 \times 38 400 = 44 160 \]
The total cost is $44 160.

Since there are 48 payments, divide the total cost by 48.
\[ 44 160 \div 48 = 920 \]
Each payment is $920, so \( R = 920 \).
There are 12 payments a year for 4 years less the first payment already made, so \( n = 47 \).
The interest rate is 6% per annum, compounded monthly.
\[ 0.06 \div 12 = 0.005, \text{ so } i = 0.005. \]

\[
PV = R \left[ \frac{1 - (1 + i)^{-n}}{i} \right]
\]

Substitute known values:
\[
= 920 \left[ \frac{1 - (1 + 0.005)^{-47}}{0.005} \right]
\]

Simplify:
\[
= 920 \left( \frac{1 - (1.005)^{-47}}{0.005} \right)
\]
\[ \approx 38449.76 \]

Carmen must invest $38,449.76 at the time of the purchase.

**Solution 2** Graphing-Calculator Method

Change the **mode settings** to 2 decimal places. From the **Finance menu**, choose **TVM Solver**.

Enter the known values.
There are 48 monthly payments, but the first payment is made at the time of the purchase, so \( N = 48 - 1 \).
The interest rate is 6% per annum, so \( I = 6 \).
With 7% GST and 8% PST, the total cost is 115% of $38,400. There are 48 payments, including the first at the time of the purchase. Carmen wants to receive the money for the payment each month, so

\[ PMT = 1.15 \times 38400 \div 48. \]

There are 12 payments per year, so \( P/Y = 12 \).
The interest is compounded monthly, so \( C/Y = 12 \).
The payments are made at the end of each payment interval, so select END.

Move the cursor to PV to find the present value, and press **ALPHA SOLVE**. Since Carmen is paying out the present value, PV is negative.

Carmen must invest $38,449.76 at the time of the purchase.
**Example 3 Finding the Value of the Payments**

To provide an annual scholarship for 25 years, a donation of $50 000 is invested in an account for a scholarship that will start a year after the investment is made. If the money is invested at 5.5% per annum, compounded annually, how much is each scholarship?

**Solution 1 Paper-and-Pencil Method**

Use the formula for the present value of an ordinary annuity.

The donation is $50 000, so $PV = 50 000$.

There is 1 payment per year for 25 years, so $n = 25$.

The interest rate is 5.5% per annum, compounded annually, so $i = 0.055$.

$$PV = R \left[ \frac{1 - (1 + i)^{-n}}{i} \right]$$

Substitute known values: 

$$50 000 = R \left[ \frac{1 - (1 + 0.055)^{-25}}{0.055} \right]$$

Simplify:

$$50 000 = R \left[ \frac{1 - (1.055)^{-25}}{0.055} \right]$$

Solve for $R$:

$$R \approx 3727.47$$

Each scholarship is $3727.47.

**Solution 2 Graphing-Calculator Method**

Change the **mode settings** to 2 decimal places. From the **Finance menu**, choose **TVM Solver**.

Enter the known values.

There is 1 payment per year for 25 years, so $N = 25$.

The interest rate is 5.5% per annum, so $I = 5.5$.

Since the donation is paid out, $PV$ is negative. The donation is $50 000, so $PV = -50 000$.

The scholarship is annual, so $P/Y = 1$.

The interest is compounded annually, so $C/Y = 1$.

The payments are made at the end of each payment interval, so select **END**.

Move the cursor to **PMT** to find the payment, and press **ALPHA SOLVE**.

Each scholarship is $3727.47.
Key Concepts

- The present value of an annuity is the principal that must be invested now to provide a given series of equal payments at equal intervals of time.
- The present value of an annuity is a financial application of the sum of a geometric series.
- The formula for the present value, \( PV \), of an ordinary annuity is 
  \[
  PV = \frac{R}{i} \left(1 - \left(1 + \frac{i}{n}\right)^n\right)
  \]
  where \( PV \) is the present value, \( R \) is the payment made at the end of each compounding period, \( n \) is the number of compounding periods, and \( i \) is the interest rate per compounding period.

Communicate Your Understanding

1. Explain what \( n \), \( R \), and \( i \) represent in the formula for the present value of an ordinary annuity.
2. Describe how you would find how much money must be invested now at 5% per annum, compounded semi-annually, to provide 10 payments of $500 every 6 months, starting in half a year.
3. Describe how you would find 24 monthly payments resulting from an investment of $78 000 at 10% per annum, compounded monthly, with the first payment to be received a month from the time of the investment.

Practise

A

1. Jack deposited $11 718.83 into an account with an interest rate of 4.4% per annum, compounded monthly. He will receive $1000 each month, starting a month after the deposit.
   a) What is the present value?
   b) What is the compounding period?
   c) What is the interest rate per annum?
   d) What is the interest rate per compounding period?
   e) What is the payment?
2. Find the present value of each investment, if the payments start in a year. The interest rate is 4.1% per annum, compounded annually.
   a) 6 annual payments of $3000
   b) 9 annual payments of $1500
3. For each investment plan, how much must be invested today at 6.25% per annum, compounded monthly, if the first payment is made a month from today?
   a) 36 monthly payments of $250
   b) 10 monthly payments of $900
4. For each investment, how much must be deposited now to receive 12 payments of $1000?
   a) 6% per annum, compounded annually, with annual payments, starting in a year
   b) 6% per annum, compounded semi-annually, with a payment every 6 months, starting in 6 months
   c) 6% per annum, compounded quarterly, with a payment every 3 months, starting in 3 months
   d) 6% per annum, compounded monthly, with monthly payments, starting in a month

5. Consider the present values from question 4. What happens to the present value of an ordinary annuity as the length of the compounding period decreases? Explain why this happens.

6. For each investment, $25 000 is deposited in an account. How much is each payment?
   a) 8% per annum, compounded annually, with annual payments starting in a year
   b) 8% per annum, compounded semi-annually, with payments every 6 months starting in 6 months
   c) 8% per annum, compounded quarterly, with payment every 3 months starting in 3 months
   d) 8% per annum, compounded monthly, with monthly payments starting in a month

7. Consider the payments for question 6. What happens to the payments as the length of the compounding periods decreases? Explain why this happens.

8. **Camp expenses** Shelley is on a committee opening a camp next year. The committee estimates that the camp will need an annual supplement of $7000 at the beginning of each year for the first 5 years. The committee decides to invest money in a plan with an interest rate of 5.3% per annum, compounded annually, to cover the cost of supplements. How much should be invested now?

9. **Lottery** A lottery to raise funds for a hospital is advertising a $240 000 prize. The winner will receive $1000 every month for 20 years, starting a year from now.
   a) If the interest rate is 8.9% per annum, compounded annually, how much must be invested now to have the money to pay this prize?
   b) If the lottery were able to negotiate an interest rate of 9.3% per annum compounded annually, how much would be invested now?
10. **Application** When Jodi’s grandmother retired, she decided to invest some money so she would receive $10 000 every 6 months for 10 years, starting in half a year. Her investment plan pays interest at 5.9% per annum, compounded semi-annually.
   a) How much must she invest?
   b) Draw a time line to illustrate the investment. Explain how the time line supports your answer.

11. **Insurance** Cora received an insurance settlement of $80 000, which she invested at 5.2% per annum, compounded monthly, to provide a payment each month for 10 years, starting next month.
    a) How much will each payment be?
    b) How much did Cora’s insurance settlement give her altogether?

12. **University** A year before Morley started university, her parents invested $22 000 at 4.9% per annum, compounded annually. When she started university, she received the same amount of money every year for 4 years.
    a) How much was each payment?
    b) Use your research skills to find how close each payment is to the cost of tuition each year.

13. **Triathlon** Wray bought a bicycle for $2500, plus GST and PST, to compete in triathlons. He arranged to make a payment to the store at the end of every month for 2 years. The store is charging 11% interest per annum, compounded monthly.
    a) How much is each payment?
    b) How much interest is Wray paying?

14. **Making financial decisions** Josh has sold his business and plans to retire. He wants to deposit enough of the money from the sale to receive a payment of $4000 a month for 20 years, starting in a month. He is considering an investment with an interest rate of 5.4% per annum, compounded monthly.
    a) How much does he need to deposit now for this investment?
    b) What are some assumptions made in this plan?

15. **Communication** Describe the difference between your calculations for the ordinary annuities in this section and the ordinary annuities in the preceding section. Include at least one question from each section in your description.
16. **Scholarship** Marvin's graduating class raised $2198.74 to establish a fund for a scholarship of $200, starting the next year, for the student who contributed most to the school. The money is invested at 4.8% per annum, compounded annually.

a) For how many years will this scholarship be awarded?

b) Explain why this problem would be difficult to solve without a graphing calculator.

17. **Inquiry/Problem Solving** Natalie's family has decided that in 3 months, they will start depositing $160 every 3 months for 3 years. The account will earn interest at 5.7% per annum, compounded quarterly. Then, 3 months after the last deposit, they plan to withdraw money every 3 months for 24 equal payments for music lessons for Natalie. How much will each withdrawal be?

18. **Estimation** Choose one question in this section.

a) Estimate to decide whether your answer is reasonable.

b) Explain your estimation strategy to a few classmates. Then, write your opinion about whether it made sense to them.

19. **True or False** Classify each statement as true or false for the present value of an ordinary annuity. Justify your answer with an explanation or examples.

a) As the interest rate increases, but the compounding period does not change, the present value increases.

b) As the compounding period decreases, the amount of money that needs to be invested decreases.

c) As the payment received each month increases, the present value decreases.

20. Research the cost of arranging an annuity. How might this influence your decision to get an annuity?

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**Achievement Check**

<table>
<thead>
<tr>
<th>Knowledge/Understanding</th>
<th>Thinking/Inquiry/Problem Solving</th>
<th>Communication</th>
<th>Application</th>
</tr>
</thead>
</table>

Jonah signed a loan contract that requires a down payment of $1500 and payments of $200 a month for 10 years. The interest rate is 9% per annum, compounded monthly.

a) What is the cash value of the contract?

b) If Jonah missed the first 8 payments, what must he pay at the time the ninth payment is due to bring himself up to date?

c) If Jonah missed the first 8 payments, what must he pay at the time the ninth payment is due to pay off the contract completely?