Generally, people must borrow money when they purchase a car, house, or condominium, so they arrange a loan or mortgage. Loans and mortgages are agreements between a money lender and a borrower to finance a purchase. The agreement usually requires the borrower to repay the loan in equal payments at equal time intervals, with payments that include both principal and interest.

To amortize a mortgage means to repay the mortgage over a given period of time in equal payments at regular intervals. The period of time is known as the amortization period. The term of a mortgage is the length of time the mortgage agreement is in effect.

**Investigate & Inquire**

An amortization table can be created to show how much of the principal remains after each payment, and how much of each payment is interest or principal.

Suppose you borrow $10 000 at an interest rate of 6% per annum, compounded monthly, and agree to pay back the loan with equal payments at one month intervals over one year. You need to know the monthly payment and the principal that has been repaid at any time during the year.

You can use a graphing calculator to find the monthly payments for the loan and a spreadsheet to keep track of the amounts paid and owed. Change the mode settings to 2 decimal places. From the Finance menu, choose TVM Solver.

Enter the known values.
There are 12 payments a year, so \( N = 12 \).
The interest rate is 6% per annum, so \( I = 6 \).
The loan is for $10 000, so PV = 10 000.
A payment is made each month, so P/Y = 12.
The interest is compounded monthly, so C/Y = 12.
The payments are made at the end of each payment interval,
so select END.

Move the cursor to PMT to find the payment for the loan,
and press ALPHA SOLVE. Since the payments for the loan
are paid out, PMT is negative.

The payment for the loan is $860.66.

Use a spreadsheet to follow the progress of repaying the loan with an
amortization table. The following spreadsheet shows data for the $10 000
loan at the top left, and formulas for calculations with the loan. Then,
formulas are copied down into other rows.

1. a) Explain the value entered in each of these cells.
   A1    A2    A4    A8
   b) Explain the formula entered in each of these cells.
   A3    A9    B8    B9    E8    F8
   c) Make a generalization about what happens to the formulas in the cells
      listed in part b) when they are copied into other rows.
   d) Identify the cells with $ in the formulas.
   e) Make a generalization about what happens to the formulas with $ when
      they are copied into other rows.

2. The following spreadsheet shows the results of the calculations from the
   formulas. The results of the calculations are seen on a spreadsheet. You can
   see a formula for an individual cell by selecting the cell.
a) For each of the following cells, explain how the formula in the first spreadsheet results in the value in the second spreadsheet.

A3  A9  B8  B9  E8  F8
C8  C9  C10  D8  D9  D10

b) Make a generalization about how the values in cells where formulas are copied are related.

c) Make a generalization about what happens to the values in cells where formulas with $ are copied.

3. a) Create the spreadsheet from question 1. Check your results with the spreadsheet from question 2.

b) Predict for how many rows you should copy the formula for this loan. Then, copy the formula down, and compare the results with your prediction.

c) Format the cells to express amounts of money to 2 decimal places.

4. a) Choose various months in the spreadsheet, and state how much of the principal has been repaid for each month. Include the row for the 12th month.

b) Explain whether the results in your spreadsheet are what you would expect.

A mortgage is a loan secured by real estate. Generally, a mortgage is arranged to finance the purchase of property. However, a mortgage can be set up for someone to borrow money for any reason, using property as security for the loan.

By Canadian law, the interest rate on mortgages is compounded semi-annually. Although other lengths of time can be arranged, most mortgages are paid monthly. This is contrary to the investments and loans in previous sections of this chapter, where the compounding period and the payment period are the same. Because of this difference, some calculations for mortgages are different.
Suppose you arrange a mortgage of $112,500 on a house, with an interest rate of 7.5% per annum, compounded semi-annually, and agree to make equal monthly payments. If you amortize the mortgage over 25 years, this means that the amount paid each month is the amount that would pay off the loan in 25 years if the mortgage continued. Generally, mortgages are for a term such as 3, 4, or 5 years. Planning a mortgage would require knowing the monthly payments, and you would probably want to know how much of the principal, the $112,500, had been repaid at various times during the mortgage.

Use a graphing calculator to find the monthly payments for the mortgage, and the monthly interest rate. The monthly interest rate, or rate per month, is calculated differently for a mortgage than for a loan because the compounding period and the payment period of a mortgage are not the same. A spreadsheet can show amounts paid and owed.

Choose the mode settings to 2 decimal places. From the Finance menu, choose TVM Solver.

Enter the known values.
There are 12 payments a year for 25 years, so $N = 25 \times 12$.
The interest rate is 7.5% per annum, so $I = 7.5$.
The mortgage is $112,500, so $PV = 112,500$.
A payment is made each month, so $P/Y = 12$.
The interest is compounded semi-annually, so $C/Y = 2$.
The payments are made at the end of each payment interval, so select END.

Move the cursor to PMT to find the payment for the loan, and press ALPHA SOLVE. Since the payment is paid out, PMT is negative.

The monthly payment for this mortgage is $823.00.

For finding the monthly interest rate, change the mode settings to 9 decimal places. From the Finance menu, choose TVM Solver. Enter values for $1$ for 1 month of this mortgage.

Enter the known values.
The interest rate per month is being calculated, so $N = 1$.
The interest rate is 7.5% per annum, so $I = 7.5$.
Use the negative amount $1$ to find the value per dollar paid out, so $PV = -1$. 
The payments are monthly, so \( P/Y = 12 \).
The interest is compounded semi-annually, so \( C/Y = 2 \).
The payments are made at the end of each payment interval, so select \text{END}.

Move the cursor to \text{FV} to find the future value of $1, and press \text{ALPHA SOLVE}.

The future value of $1 is $1.006154524.

Since the future value is the amount with compound interest for the present value, or investment of $1, \( A = P + i \), where \( i \) represents the monthly interest rate.

Amount equals principal plus interest. \( A = P + i \)

Substitute known values: \( 1.006154524 = 1 + i \)

Isolate \( i \):
\[
i = 0.006154524
\]

The interest rate per month is 0.006154524, or 0.6154524\%.

Use a spreadsheet to follow the progress of repaying the mortgage with an amortization table. The following spreadsheet shows data for the $112,500 mortgage at the top left, and formulas for calculations for the mortgage. Then, formulas are copied down into other rows.

5. a) Explain the value entered in each of these cells.
A1     A2     A4     A8
b) For each of these cells, explain the formula and predict the value that it will calculate.
A9     B8     B9     E8     F8
c) Make a generalization about what happens to the formulas in cells listed in part b) where they are copied into other rows.
d) Make a prediction about what will happen for the values in cells where formulas listed in part b) are copied.
e) Make a generalization about what happens to the formulas with $ as they are copied into other rows.
f) Make a prediction about what will happen to the values in cells where formulas with $ are copied.

6. a) Create the spreadsheet shown on page 548.
b) Predict for how many rows you should copy the formulas for this mortgage. Then, copy the formulas down, and compare the results with your prediction.
c) Format the cells to express amounts of money to 2 decimal places.

7. a) Choose various months in the spreadsheet, and state how much of the principal has been repaid. Include the row for 25 years.
b) How close is the last value in the Remaining Principal column to $0.00? Because the monthly payment of $823.00 was obtained by rounding to the nearest cent, the values rounded make a difference over 25 years. Explain why.
c) What do you think would happen in the spreadsheet if all the decimal values for the monthly payment were used?

8. Explain whether the results in your spreadsheet are what you would expect.

**Example 1  Finding the Length of Time to Pay Off a Loan**

Gurjeet is spending $1875.25 on furniture for her apartment. Her uncle offers to finance this purchase, charging an interest rate of 4% per annum, compounded monthly. Gurjeet agrees to pay $125 per month for the loan payment.

a) Use a spreadsheet to find how long it takes Gurjeet to pay off the loan.
b) What two methods can you use to find Gurjeet's final payment? How can you explain the difference in the results obtained by these two methods?

**Solution**

a) Enter the data in a spreadsheet.
Gurjeet's loan is $1875.25, so enter 1875.25 in cell A1.
The interest rate per annum is 4%, so enter 0.04 in cell A2.
The interest rate per month is the annual interest rate divided by the number of months in a year.
Gurjeet’s monthly payment is $125, so enter 125.00 in cell A4.

Use formulas shown in the spreadsheets in the Investigate & Inquire. Copy the formulas down until the rows show that the principal is repaid.

The row for 16 months is the last row that shows principal owing. The loan will be repaid in 16 months.

b) One method is to find the sum of the principal and the interest on this principal in the 16th month. So, the final payment is the sum of the values in cells B23 and C23.

51.84 + 0.17 = 52.01

Using the sum of the principal and the interest on this principal for the 16th month, Gurjeet’s final payment is $52.01.

Another method is to find the difference between Gurjeet’s usual payment of $125.00 and the negative remaining principal for the 16th month.

125.00 − 72.98 = 52.02

Using the difference between Gurjeet’s usual payment and the negative remaining principal for the 16th month, Gurjeet’s final payment is $52.02.

The difference in the results obtained by these two methods occurs because of rounding for calculations in the spreadsheet.

**Example 2 Comparing the Effects of Changing the Amortization Period**

Robyn and Jonathon are working with a bank manager to arrange a mortgage of $200 000 to buy their new home. The bank will charge interest on their mortgage at 8.36% per annum, compounded semi-annually.

a) For a mortgage amortized over 25 years, find

i) the monthly payment

ii) the monthly interest rate

iii) the total interest

iv) the total amount of the payments
b) With the mortgage amortized over 25 years, how much do Robyn and Jonathon owe after 15 years?

c) For a mortgage amortized over 20 years, find the
   i) monthly payments
   ii) monthly interest rate
   iii) total interest
   iv) total amount of the payments

d) Compare Robyn and Jonathon’s cost of amortizing their mortgage over 25 years with the cost of amortizing over 20 years.

SOLUTION

a) i) Use a graphing calculator to find the monthly payments. Change the mode settings to 2 decimal places. From the Finance menu, choose TVM Solver.

Enter the known values.

There are 12 payments a year for 25 years, so \( N = 25 \times 12 \).

The interest rate is 8.36% per annum, so \( I = 8.36 \).

The mortgage is $200 000, so \( PV = 200 \, 000 \).

A payment is made each month, so \( P/Y = 12 \).

The interest is compounded semi-annually, so \( C/Y = 2 \).

The payments are made at the end of each payment interval, so select END.

Move the cursor to PMT to find the mortgage payment, and press ALPHA SOLVE. Since the mortgage payment is paid out, PMT is negative.

The monthly payments are $1572.63.

ii) Use a graphing calculator to find the interest rate per month. Change the mode settings to 9 decimal places. From the Finance menu, choose TVM Solver.

Enter the known values.

The interest rate per month is being calculated, so \( N = 1 \).

The interest rate is 8.36% per annum, so \( I = 8.36 \).

Use the negative amount $1 to find the value per dollar paid out, so \( PV = -1 \).

A payment is made each month, so \( P/Y = 12 \).

The interest is compounded semi-annually, so \( C/Y = 2 \).

The payments are made at the end of each payment interval, so select END.

Move the cursor to FV to find the future value of $1, and press ALPHA SOLVE.

The future value of $1 is $1.006 848 341.
Since the future value is the compound amount for the present value, or the investment for $1, \( A = P + i \), where \( i \) represents the monthly interest rate.

\[
A = P + i
\]

**Substitute known values:** \( 1.006 \ 848 \ 341 = 1 + i \)

**Solve for \( i \):** \( i = 0.006 \ 848 \ 341 \)

The monthly interest rate is 0.006 848 341, or 0.684 834 1%.

**iii) Enter the data in cells A1 to A4. Use formulas shown in the spreadsheet in the Investigate & Inquire. Copy the formulas down until the rows show the payments for 25 years.**

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>200000.00</td>
<td>Mortgage ($)</td>
<td>Robyn and Jonathon's 25-year mortgage</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0830</td>
<td>Rate/Year</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.006848341</td>
<td>Rate/Month</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1572.63</td>
<td>Payment ($)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Month</td>
<td>Principal ($)</td>
<td>Interest ($)</td>
<td>Payment ($)</td>
<td>Principal Reduction ($)</td>
<td>Remaining Principal ($)</td>
</tr>
<tr>
<td>1</td>
<td>200000.00</td>
<td>1369.67</td>
<td>1572.63</td>
<td>202.96</td>
<td>199797.04</td>
</tr>
<tr>
<td>2</td>
<td>199797.04</td>
<td>1368.26</td>
<td>1572.63</td>
<td>204.35</td>
<td>199592.69</td>
</tr>
<tr>
<td>3</td>
<td>199592.69</td>
<td>1366.86</td>
<td>1572.63</td>
<td>205.75</td>
<td>199388.94</td>
</tr>
<tr>
<td>4</td>
<td>199388.94</td>
<td>1365.47</td>
<td>1572.63</td>
<td>207.15</td>
<td>199185.09</td>
</tr>
<tr>
<td>5</td>
<td>199185.09</td>
<td>1364.08</td>
<td>1572.63</td>
<td>208.55</td>
<td>198981.54</td>
</tr>
</tbody>
</table>

In cell C308, enter the formula for the sum of the values in cells C8 to C307. The sum in cell C308 shows that the total interest is $271 786.15.

**iv) In cell D308, enter the formula for the sum of the values in cells D8 to D307. The sum in cell D308 shows that the total amount of the payments is $471 789.00.**

The negative value in cell F307 indicates that Robyn and Jonathon overpaid by $2.85. If, however, the full decimal accuracy were entered the final remaining principal would be $0.00.

Variations because of rounding are also found by entering the formula for the sum of the values in cells E8 to E307 into cell E308. The sum for the principal reduction is $2.85 greater than the mortgage of $200 000.
b) **Spreadsheet Method**

Find the number of months in 15 years.

\[ 15 \times 12 = 180 \]

The row in the amortization table for 180 months shows the remaining principal $128,394.92.

After 15 years, Robyn and Jonathon owe $128,394.92.

**Graphing-Calculator Method**

Change the **mode settings** to 2 decimal places. From the **Finance menu**, choose **TVM Solver**.

Enter the known values.

After 15 years, 10 years of the 25-year amortization period remain with 12 payments per year, so \( N = 10 \times 12 \).

The interest rate is 8.36% per annum, so \( I = 8.36 \).

Since the mortgage payment is paid out, \( PMT \) is negative. The monthly payments are $1572.63, so \( PMT = -1572.63 \).

A payment is made each month, so \( P/Y = 12 \).

The interest is compounded semi-annually, so \( C/Y = 2 \).

The payments are made at the end of each payment interval, so select **END**.

Move the cursor to **PV** to find the present value, and press **ALPHA SOLVE**.

After 15 years, Robyn and Jonathon owe $128,396.17.

c) i) Use a graphing calculator to find the monthly payments. Change the **mode settings** to 2 decimal places. From the **Finance menu**, choose **TVM Solver**.

Enter the known values.

There are 12 payments a year for 20 years, so \( N = 20 \times 12 \).

The interest rate is 8.36% per annum, so \( I = 8.36 \).

The mortgage is $200,000, so \( PV = 200,000 \).

A payment is made each month, so \( P/Y = 12 \).

The interest is compounded semi-annually, so \( C/Y = 2 \).

The payments are made at the end of each payment interval, so select **END**.

Move the cursor to **PMT** to find the mortgage payment, and press **ALPHA SOLVE**. Since the mortgage payment is paid out, \( PMT \) is negative.

The monthly payments are $1700.12.
ii) Since the monthly rate for amortizing the mortgage over 20 years is the same as for amortizing the mortgage over 25 years, use the monthly interest rate from part a) ii).

So, the monthly interest rate is 0.006848341, or 0.6848341%.

iii) Create a spreadsheet like this. Enter the formulas for sums of columns in row 248.

<table>
<thead>
<tr>
<th>Month</th>
<th>Principal ($)</th>
<th>Interest ($)</th>
<th>Payment ($)</th>
<th>Principal Reduction ($)</th>
<th>Remaining Principal ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>200000.00</td>
<td>1995.67</td>
<td>1700.12</td>
<td>330.45</td>
<td>196699.55</td>
</tr>
<tr>
<td>9</td>
<td>196699.55</td>
<td>1970.41</td>
<td>1700.12</td>
<td>332.71</td>
<td>193368.83</td>
</tr>
<tr>
<td>10</td>
<td>193368.83</td>
<td>1965.12</td>
<td>1700.12</td>
<td>334.99</td>
<td>190010.84</td>
</tr>
</tbody>
</table>

The sum in cell C248 shows that the total interest is $208,027.28.

iv) The sum in cell D248 shows that the total amount of the payments is $408,028.80.

The negative value in cell F247 indicates that Robyn and Jonathon overpaid by $1.52. If the full decimal accuracy were entered the final remaining principal would be $0.00.

Variations because of rounding are also found by entering the formula for the sum of the values in cells E8 to E247 into cell E248. The sum for the principal reduction is $1.52 greater than the mortgage of $200,000.

d) For a mortgage amortized over 25 years, the sum of the payments is the value in cell D308 of the spreadsheet in part a). For a mortgage amortized over 20 years, the sum of the payments is the value in cell D248 of the spreadsheet in part c).

471,789.00 − 408,028.80 = 63,760.20

Robyn and Jonathan would save $63,760.20 by amortizing the mortgage over 20 years instead of 25 years.
Key Concepts

- A mortgage is a loan secured by real estate.
- An amortization table shows the progress of repaying a loan or a mortgage over a given period of time in equal payments at regular intervals. The payments generally include principal and interest.
- The term of a mortgage is the length of time the mortgage agreement is in effect.
- Interest on a mortgage is calculated differently than on a loan. In Canada, the interest on a mortgage is compounded semi-annually. For a loan, interest can be compounded monthly and calculated monthly.

Communicate Your Understanding

1. Explain the advantages of using spreadsheets to develop amortization schedules.
2. Describe how you would use a graphing calculator to find the monthly interest rate equivalent to the interest rate 6% per annum, compounded semi-annually.
3. Describe how you would set up an amortization table for a mortgage of $95 000, amortized over 25 years with a monthly interest rate of 0.776 438 3%.
4. Explain how you decide
   a) how many decimal digits to format for each cell
   b) how far to copy the formulas into the rows

Practise

A

1. Use a graphing calculator to find an equivalent monthly rate for each rate per annum, compounded semi-annually.
   a) 6%    b) 10%    c) 5.5%    d) 20.46%

2. The equivalent monthly rates calculated in question 1 are for mortgages. Why are these calculations necessary?

3. Use a graphing calculator to find the monthly payment for each loan.
   a) a car loan of $19 275 at 6% per annum, compounded monthly, with monthly payments for 5 years
   b) a personal loan of $12 000 at 9% per annum, compounded annually, with yearly payments for 15 years
   c) a mortgage of $275 000 with interest charged at 6.95% per annum, compounded semi-annually, with monthly payments for 15 years

4. Use a graphing calculator to determine the monthly interest rate for a loan at each rate per annum, compounded annually.
   a) 10%    b) 7.5%    c) 5.25%
   d) 3%    e) 18%
**Apply, Solve, Communicate**

5. **Computer systems** A company is selling computer systems, including a printer, scanner, and software for $3000. The cost is financed at a rate of 10% per annum, compounded monthly, paid off at $150 per month.
   a) How long does it take to pay off the loan?
   b) What amount is the final payment?

6. **A store** Kathleen is talking to a bank about a $134,000 mortgage for a store she is buying. The bank is offering an interest rate of 9.75%, with a 25-year amortization plan.
   a) Use a graphing calculator to find Kathleen’s monthly interest rate.
   b) Use a spreadsheet to find Kathleen’s mortgage payments.
   c) How much would Kathleen owe after 4 years?

7. **A new home** Andrea is ready to sign a mortgage worth $140,000 on her new home. Her bank manager says the interest rate will be 7%, on a mortgage repaid over 20 years.
   a) Use a graphing calculator to find the monthly interest rate and payment.
   b) Use a spreadsheet to create an amortization table.
   c) If the bank increases her interest rate to 7.25%, but allows the same payment, how much would she owe on the mortgage after 20 years?
   d) If the interest rate is changed to 7.25%, but the payment remains the same, how long would it take her to pay off the mortgage?

8. **Town house** The Khan family has a mortgage of $160,000 to finance their town house. The bank is charging interest at a rate of 6.25% amortized over 25 years.
   a) Find the Khan family’s monthly interest rate and payment.
   b) What percent of each payment is used to pay interest
      i) for the first payment?
      ii) for the second payment?
      iii) for the third payment?
   c) What trend do you notice in the percent of the payments used to pay interest? Explain whether this trend makes sense.

9. **Application** The Vandenberghe family are refinancing their mortgage of $155,000 with a plan for repaying the mortgage over 25 years. The bank is charging interest at 9%. Use a spreadsheet to find the first time the value in the Principal Reduction column is greater than the value in the Interest column. What do these values mean about the mortgage?
10. Consider the spreadsheet at the right.
   a) How much is owed on the loan after 4 months?
   b) How long does it take to repay at least half the principal?
   c) Create a spreadsheet so the amount borrowed will be repaid after
      i) 6 months  ii) 2 years
   d) Explain your method for part c) to a few classmates. Then, describe their responses.

   ![Spreadsheet Image](image)

11. **Comparing** The Garcias have just taken out a mortgage for $125 000. Their bank charges interest at 8.5% for a mortgage amortized over 25 years. Their new neighbours, the Picards, are arranging a loan of $130 000, amortized over 20 years. The interest at their bank for this loan is 7% per annum, compounded annually.
   a) Predict who has a greater monthly payment. Justify your prediction.
   b) Use a spreadsheet to find who has a greater monthly payment and what the difference is between the payments.
   c) Compare your spreadsheet calculations with your predictions.

12. **Application** When Pat graduated from university, he found a job in Thunder Bay. He negotiated a mortgage of $95 000 for a condominium. The bank's interest rate was 7.75% for a mortgage amortized over 25 years with a 5-year term. At the end of the term, he renegotiated his mortgage. Since the interest rate had decreased to 7.25%, he decided to reduce the remaining time from 20 years to 15 years.
   a) Use a spreadsheet to find his new monthly payment.
   b) Change your spreadsheet to show the amortization table if the time had remained at 20 years.
   c) What is the difference between the payments on the spreadsheets for parts a) and b)?
   d) What is the difference between the sum of the payments for parts a) and b)?
13. **Farm** After the Singhs made a down payment on a small farm, they took out a mortgage of $225 000, amortized over 25 years at 6.75%.
   a) How much interest will be paid during the sixth year of their repayment schedule?
   b) How much of the principal will remain after the sixth year?
   c) Research mortgage rates for two different times, a decade apart. Repeat parts a) and b) for each of your researched rates.

14. **Buying a jeep** After Collin pays the down payment on a jeep, he has a debt of $15 000. The dealership offers a loan amortized over 5 years with an interest rate of 9% per annum, compounded monthly. If Collin can afford to pay only $300 per month, should he buy this car? Use spreadsheet calculations to support your answer.

15. **Trial and error** Design a spreadsheet to show the amortization of a loan of $20 000 with interest calculated at 8.5% per annum, compounded monthly. Monthly payments are made over 3 years. Instead of using a graphing calculator to find the monthly payment, use a trial-and-error method to find the exact payment for the loan, correct to the nearest hundredth of a cent.

16. **Scatter plot** The Soligos are planning a mortgage of $75 000 for their condominium. Their bank charges interest at 8.5% for a mortgage amortized over 15 years.
   a) Use a spreadsheet to create an amortization table.
   b) Use the values in the table to find the amount they owe after each year. Copy and complete the following table with these values.
   c) Enter the values for Year into L1 of a graphing calculator and the values for Amount owing into L2.
   d) Create a scatter plot for the data.
   e) Determine the equation of a curve of best fit for the data.
   f) Use this equation to determine the amount owing after
      i) 4 years
      ii) 3 months
   g) Determine how close the value after 4 years is to the value in the spreadsheet.

17. **Communication** a) What are the advantages of a 15-year mortgage? What are the disadvantages?
   b) What are the advantages of a 25-year mortgage? What are the disadvantages?