### 8.2 Equations of Loci

A locus is a set of points defined by a rule or condition. For example, if a dog is attached by a 10-m leash to a post in the middle of a large yard, then the locus of the farthest points that the dog can reach is a circle with a radius of 10 m.

In analytic geometry, equations can be used to describe loci.

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**INVESTIGATE & INQUIRE**

1. **a)** Draw a diagram that shows six points that are exactly 2 units from the x-axis.
   **b)** How would you change your diagram to show all the points that are exactly 2 units from the x-axis?
   **c)** Write an equation, or equations, to describe the points in part b), that is, the locus of points that are exactly 2 units from the x-axis.

2. **a)** How are the lines \( y = x + 5 \) and \( y = x - 7 \) related?
   **b)** Draw a diagram to show the two lines and the locus of points that are equidistant from both of them.
   **c)** Write an equation to describe this locus.

3. **a)** Use a diagram to represent the locus of points that are equidistant from both axes.
   **b)** Write an equation, or equations, to describe this locus.

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To determine an equation to represent a described locus, use the following steps.

**Step 1** Construct a diagram showing the given information.

**Step 2** Locate several points that satisfy the rule or condition.

**Step 3** Draw a curve or line using the located points.

**Step 4** Write an equation for the locus.
**Example 1 Constructing a Locus**

For $\angle ABC = 60^\circ$, construct a geometric model to represent the locus of points in the interior of the angle that are equidistant from the sides.

**Solution**

Step 1 Construct $\angle ABC = 60^\circ$.

Step 2 Use two clear plastic rulers to locate several points whose perpendicular distances from each side of the angle are equal.

Step 3 Draw a ray through the points.

Note that, in Example 1, the locus of points in the interior of the angle that are equidistant from the sides is the angle bisector.
**Example 2** Determining an Equation for a Locus

**a)** Determine an equation to represent the locus of points equidistant from the points \(A(4, 3)\) and \(B(-2, 1)\).

**b)** Determine the relationship between the line segment \(AB\) and the locus of points equidistant from the points \(A(4, 3)\) and \(B(-2, 1)\).

**Solution**

**a)** Step 1 Plot points \(A(4, 3)\) and \(B(-2, 1)\) on a coordinate grid.

Step 2 Locate several points that are equidistant from \(A\) and \(B\).

Step 3 Draw a line through the points.
Step 4  To determine an equation for the locus of points, use the formula for
the length of a line segment, \( l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \).
Let \( P(x, y) \) be any point equidistant from \( A(4, 3) \) and \( B(-2, 1) \).
So, \( PA = PB \).
\[
PA = \sqrt{(x - 4)^2 + (y - 3)^2}
\]
\[
PB = \sqrt{(x + 2)^2 + (y - 1)^2}
\]
Since \( PA = PB \),
\[
(x - 4)^2 + (y - 3)^2 = (x + 2)^2 + (y - 1)^2
\]
Square both sides:
\[
(x - 4)^2 + (y - 3)^2 = (x + 2)^2 + (y - 1)^2
\]
Expand:
\[
x^2 - 8x + 16 + y^2 - 6y + 9 = x^2 + 4x + 4 + y^2 - 2y + 1
\]
Simplify:
\[-12x - 4y + 20 = 0
\]
Divide both sides by -4: 
\[
3x + y - 5 = 0
\]
An equation of the locus is \( 3x + y - 5 = 0 \).

b) The slope of the line segment joining \( A(4, 3) \) and \( B(-2, 1) \) is
\[
m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}
\]
\[
= \frac{1 - 3}{-2 - 4}
\]
\[
= \frac{-2}{-6}
\]
\[
= \frac{1}{3}
\]
The slope of the locus can be found by writing the equation from part a) in
the form \( y = mx + b \), where \( m \) is the slope.
\[
3x + y - 5 = 0
\]
\[
y = -3x + 5
\]
So, the slope of the locus is \(-3\).
Since the slopes \( \frac{1}{3} \) and \(-3\) are negative reciprocals, the locus and the line
segment \( AB \) are perpendicular.
The midpoint of the line segment joining \( A(4, 3) \) and \( B(-2, 1) \) can be
found using the midpoint formula, \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \).
Key Concepts

• A locus is a set of points defined by a rule or condition.

• To determine an equation to represent a described locus, use the following steps.
  Step 1 Construct a diagram showing the given information.
  Step 2 Locate several points that satisfy the rule or condition.
  Step 3 Draw a curve or line using the located points.
  Step 4 Write the equation.

Communicate Your Understanding

1. Write an equation to represent the locus of points that are 7 units from the point (0, 0) on a coordinate grid.
2. Does a linear equation represent the locus of points that are equidistant from the vertices of a rectangle? Explain.

Practise

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1. Sketch and describe the locus of points in the plane that are
   a) in the interior of a right angle and equidistant from the sides
   b) equidistant from two parallel lines 6 cm apart
   c) n units away from a given line l
   d) equidistant from two concentric circles, one with radius 6 cm and the other with radius 4 cm
   e) equidistant from the vertices of a square

2. Determine an equation that represents the locus of points equidistant from x = −2 and x = 4.
3. a) How are the lines \( y = 2x + 1 \) and \( y = 2x - 3 \) related?  
b) Determine an equation for the locus of points equidistant from the two lines.

4. Determine an equation for the locus of points equidistant from each pair of points.  
a) \((4, -3)\) and \((2, -5)\)  
b) \((2, 4)\) and \((5, -2)\)  
c) \((-3, 5)\) and \((2, -1)\)

5. a) On a coordinate grid, construct two circles, one with radius 3 cm, the other with radius 7 cm, and both with centre \((0, 0)\).  
b) Determine an equation to represent the locus of points equidistant from the two circles.

**Apply, Solve, Communicate**

6. **Ambulance station** Two hospitals are located at \((4, -1)\) and \((3, 7)\). An ambulance station is to be built such that it is equidistant from the two hospitals. Determine an equation for the locus of points equidistant from the two hospitals.

7. **Home purchase** Andrea is purchasing a new home. She would like to be close to downtown and to her office, and the same distance from them. City Hall is downtown and is located at \((-1, -3)\). Andrea's office is located at \((4, -5)\). Find the equation of the locus of points equidistant from downtown and Andrea's office.

8. **Highway driving** Sketch the locus of points traced by each of the following on a car being driven down a highway.  
a) the centre of a wheel  
b) a point on the outer edge of a tire

9. a) Verify that the points \((-3, 4)\) and \((4, 3)\) lie on the circle \(x^2 + y^2 = 25\).  
b) Determine an equation for the locus of points equidistant from the points \((-3, 4)\) and \((4, 3)\).  
c) Determine the relationship between the centre of the circle and the locus of points equidistant from the points \((-3, 4)\) and \((4, 3)\).

10. **Inquiry/Problem Solving** Sketch the set of ordered pairs. Then, write an equation of a locus that all the points in each set might satisfy.  
a) \{(-5, 0), (5, 0), (0, -5), (0, 5)\}  
b) \{(0, 0), (-1, 1), (1, 1), (-2, 4), (2, 4), (-3, 9), (3, 9)\}  
c) \{(0, 0), (1, 1), (4, 2), (9, 3), (16, 4), (25, 5)\}
11. Determine an equation, or equations, to represent the locus of points equidistant from each pair of lines.
   a) \( y = x \) and \( y = -x \)  
   b) \( y = 2x + 2 \) and \( y = -2x + 2 \)  
   c) \( y = 2x \) and \( y = 0.5x \)

12. Determine an equation for the locus of points equidistant from each pair of graphs.
   a) \( y = -\sqrt{x} \) and \( y = \sqrt{x} \)  
   b) \( y = \sqrt{x} + 4 \) and \( y = -\sqrt{x} - 6 \)

13. Determine an equation for the locus of points equidistant from the points on the graph of \( y = 2(x - 3)^2 + 2 \).

14. **Flower bed**  
The outside edge of a fountain is the locus of points 2 m from the centre. The outside edge of a flower bed is the locus of points 3 m from the centre of the fountain. There is no gap between the flower bed and the fountain. Sketch the flower bed and calculate its area.

15. Describe and sketch the locus of points in the plane that are 13 units from the origin and 12 units from the \( y \)-axis.

16. **Communication**  
Sketch each of the following loci in the plane. Explain how your sketch represents each locus.
   a) the points 2 cm or less from a given line  
   b) the points at least 3 cm from a given point  
   c) the points from 2 cm to 5 cm from a given point

17. **Application**  
Loci can be described in three dimensions. Describe each of the following loci in two dimensions and in three dimensions.
   a) the locus of points 3 cm from a given point  
   b) the locus of points 2 cm from a given line  
   c) the locus of points equidistant from two given points  
   d) the locus of points equidistant from two parallel lines