8.3 Technology: Loci and Conics

The diagram shows a double cone. The two cones have one point in common.

The intersection of a double cone and a plane is called a conic section or a conic.

The circle, ellipse, parabola, and hyperbola are the cross sections that can be formed when a double cone is sliced by a plane. The different cross sections are formed by changing the angle and location of the plane.

Circle  Ellipse  Parabola  Hyperbola

In this section, the locus definitions of an ellipse, a hyperbola, and a parabola will be used to construct geometric models of these conics using The Geometer’s Sketchpad®.

**The Ellipse**

An ellipse is the set or locus of points \( P \) in the plane such that the sum of the distances from \( P \) to two fixed points, \( F_1 \) and \( F_2 \), is a constant.

\[ F_1P + F_2P = k \]

One method of constructing this locus using The Geometer’s Sketchpad® is described in the following steps.

1. Construct a line segment in the lower portion of the screen. Label the endpoints \( F_1 \) and \( F_2 \).

2. Construct line segment \( CD \) in the upper portion of the screen. \( CD \) must be longer than \( F_1F_2 \).
3. With line segment CD selected, choose \textit{Point on Object} from the \textit{Construct} menu. Label this point E.

4. Select points C and E. From the \textit{Construct} menu, choose \textit{Segment}.

5. Select points D and E. From the \textit{Construct} menu, choose \textit{Segment}.

6. Select point F1 and segment CE. From the \textit{Construct} menu, choose \textit{Circle By Center And Radius}.

7. Select point F2 and segment DE. From the \textit{Construct} menu, choose \textit{Circle By Center And Radius}. If necessary, drag point E so that the two circles intersect.

8. Select the two circles. From the \textit{Construct} menu, choose \textit{Point At Intersection}. Label the points P1 and P2.

9. Select F1 and P1. From the \textit{Construct} menu, choose \textit{Segment}.

10. Select F2 and P1. From the \textit{Construct} menu, choose \textit{Segment}.

11. Select line segment CD. From the \textit{Display} menu, choose \textit{Hide Segment}.

12. Select line segment CE, line segment F1P1, and the circle with centre F1. Right click and choose a colour to be applied to these three objects.

13. Select line segment ED, line segment F2P1, and the circle with centre F2. Right click and choose a different colour to be applied to these three objects.

14. Select points P1 and P2. From the \textit{Display} menu, choose \textit{Trace Points}. Drag point E along line segment CD to generate the locus of points. Describe the resulting shape.
15. Explain why the sum of $F_1P_1 + F_2P_1 = CD$. Hint: Check the colour used for $F_1P_1$ and $CE$, and then the colour used for $F_2P_1$ and $DE$.

16. Test the construction by changing the length of the line segment joining $F_1$ and $F_2$. Describe how you would change the length of this line segment in order to get ellipses that are wider and narrower.

17. **Extension**  
   
   a) Select point $E$ and line segment $CD$. You may need to choose **Show All Hidden** from the **Display** menu. From the **Edit** menu, choose **Action Button**. Choose the **Animation** sub-option. Change the first drop box to “once” and the third to “slowly.” Click on Animate to close the window. Double click on **Animate** to show an ellipse. Experiment with the placement of points $F_1$ and $F_2$ and the length of line segment $CD$ to obtain wider and narrower ellipses.

   b) From the **Graph** menu, choose **Show Grid**. Place $F_1$ and $F_2$ on the x-axis equal distances from the origin. When the construction is completed, what is the distance between the points of intersection of the locus and the x-axis? How is this distance related to the length of line segment $CD$?

   c) Select point $P_1$ and point $P_2$. From the **Edit** menu, choose **Trace Points**. This will remove the trace feature. Delete the Animate button. Select point $P_1$ and point $E$ in that order. From the **Construct** menu, choose **Locus**. Repeat for $P_2$ and point $E$. The result is a locus of points for each of $P_1$ and $P_2$. What happens to this locus of points if you drag point $F_1$?

**The Hyperbola**

The **absolute value** of a real number is its distance from zero on a real number line. Since distance is always positive, the absolute value of a number is always positive.

For a real number represented by $x$, the absolute value is written $|x|$, which means the positive value of $x$. 

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The absolute value of \(-2\) is 2, or \(|-2| = 2\).

Similarly \(|4 - 9| = |-5|\)
\[= 5\]
And \(|9 - 4| = |5|\)
\[= 5\]

A hyperbola is the set or locus of points \(P\) in the plane such that the absolute value of the difference of the distances from \(P\) to two fixed points, \(F_1\) and \(F_2\), is a constant.

\[|F_1P - F_2P| = k\]

One method of constructing this locus using The Geometer's Sketchpad® is described in the following steps.

18. Change the Line Tool to line and construct a line through points \(A\) and \(B\) near the top of the screen. Place points \(A\) and \(B\) as close to the centre of the top of the screen as possible.

19. From the Construct menu, choose Point On Object. Label this point \(C\). If necessary, move \(C\) so that it is not between \(A\) and \(B\).

20. Change the Line Tool to line segment. Near the bottom of your screen, construct a line segment. Name the endpoints \(F_1\) and \(F_2\).

21. Measure the lengths of \(AB\) and \(F_1F_2\). If necessary drag \(A\) or \(B\) so that \(AB\) is the shorter segment.

22. Construct line segments \(AC\) and \(BC\). The difference between the lengths of these two line segments will always equal the length of line segment \(AB\) and will serve as the constant for the locus.

23. Select point \(F_1\) and line segment \(AC\). From the Construct menu, choose Circle By Center And Radius.

24. Select point \(F_2\) and line segment \(BC\). From the Construct menu, choose Circle By Center And Radius.

25. Select the two circles. From the Construct menu, choose Point At Intersection. Label these two points \(P_1\) and \(P_2\).

26. Select points \(P_1\) and \(F_1\). From the Construct menu, choose Segment.

27. Select points \(P_1\) and \(F_2\). From the Construct menu, choose Segment.
28. Select and measure the line segments AC, BC, P1F1, and P1F2. From the **Measure** menu, choose **Calculate**. Use the calculator to find \(|m(AC) - m(BC)| \) and \(|m(P1F1) - m(P1F2)|\).

29. Drag point C along the line through A and B. Notice which segments and calculated values are equal.

30. Select P1 and P2. From the **Display** menu, choose **Trace Points**.

31. Select the two circles. From the **Display** menu, choose **Hide Circles**.

32. Drag point C along the line through A and B. For the best effect, start at one side of the screen and drag point C to the other side slowly.

33. Describe the locus of points generated by this construction.

34. **Extension**
   a) Change the length of line segment F1F2. How can you make the curves flatten? Can you get a similar effect by changing the length of line segment AB?
   b) Select point C and the line through A and B. From the **Edit** menu, choose **Action Button**. Choose the **Animation** sub-option. Change the first drop box to “once” and the third drop box to “slowly.” Click on Animate. Double click on ✝️ **Animate**.
   c) Select Point P1 and point P2. From the **Edit** menu, choose **Trace Points**. This will remove the trace feature. Delete the Animate button. Select point P1 and point C in that order. From the **Construct** menu, choose **Locus**. Repeat for P2 and point C. The result is a locus of points for each of P1 and P2. What happens to this locus of points if you drag point F1? What happens when the length of F1F2 is shorter than the length of AB?
The Parabola

A parabola is the set or locus of points P in the plane such that the distance from P to a fixed point F equals the distance from P to a fixed line l.

\[ PF = PD \]

The fixed point F is called the focus. The fixed line l is called the directrix.

One method of constructing this locus using The Geometer's Sketchpad® is described in the following steps.

35. Construct a line AB near the bottom of the screen. Label this line d for directrix.
36. Construct a point not on line d. Label it F, for focus.
37. Select point A and line d. From the Construct menu, choose Perpendicular Line.
38. With this perpendicular line selected, choose Point On Object from the Construct menu. Label this point D.
39. Select points A and D. Change the line tool to a segment tool. From the Construct menu, choose Segment.
40. Select point F and segment AD. From the Construct menu, choose Circle By Center And Radius. This creates a set of points a fixed distance from the focus.
41. We now create a set of points that are the same distance from line d. Select points A and D in that order. From the Transform menu, choose Mark Vector “A→D”.
42. Select line d. From the Transform menu, choose Translate... . Choose the option By Marked Vector. Press O K.
43. Select this new line through D and the circle. From the Construct menu, choose Point At Intersection. Label these points P and Q.
44. With points P and Q selected, choose Trace Points from the Display menu.
45. Select the circle and the line through D and P. From the Display menu, choose Hide Objects.
46. Drag point D back and forth along line AD. The path of the points P and Q represents the locus of points that are equidistant from point F and line d.

47. Describe the locus of points you constructed.

48. To test this construction, place point F at different positions. Describe the locus when the focus, point F, is closer to the directrix, line AB. Describe the locus when point F is farther away.

49. **Extension**  
   a) To start with a grid, choose **Show Grid** from the **Graph** menu. Construct the directrix as a line parallel to and below the x-axis. Place point F on the y-axis the same number of units above the origin as the directrix is below the x-axis. When the construction is complete, where is the vertex of the parabola?  
   b) Select point D and line AD. From the **Edit** menu, choose **Action Button**. Choose the **Animation** sub-option. Change the first drop box to "once" and the third drop box to "slowly." Click on Animate. Double click to get the locus of points.  
   c) Select points P and Q. From the **Edit** menu, choose **Trace Points**. This will remove the tracing of points P and Q. Click on points P and D. From the **Construct** menu, choose **Locus**. A locus of points will be created. Then select points Q and D. Again choose **Locus** from the **Construct** menu to complete the shape on the screen. Drag point F to new locations and watch the locus of points change. Describe the effect that the position of F has on the locus.