The CF-18 Hornet is a supersonic jet flown in Canada. It has a maximum speed of Mach 1.8. The speed of sound is Mach 1. When a plane like the Hornet breaks the sound barrier, it produces a shock wave in the shape of a cone. For a plane flying parallel to the ground, the shock wave, or sonic boom, intersects the ground in a branch of a hyperbola.

The standard form of the equation of a conic provides a convenient way to identify the conic and sketch the graph. The equation of a conic can also be written in the form
\[ ax^2 + by^2 + 2gx + 2fy + c = 0. \]

For example, an equation in standard form for a hyperbola is
\[ \frac{(x + 2)^2}{9} - \frac{(y - 1)^2}{4} = 1. \]

Multiply both sides by 36:
\[ 4(x + 2)^2 - 9(y - 1)^2 = 36. \]
Expand and simplify:
\[ 4(x^2 + 4x + 4) - 9(y^2 - 2y + 1) = 36, \]
\[ 4x^2 + 16x + 16 - 9y^2 + 18y - 9 = 36, \]
\[ 4x^2 - 9y^2 + 16x + 18y - 29 = 0. \]

In general, \( ax^2 + by^2 + 2gx + 2fy + c = 0 \) is a quadratic equation when \( a \) and \( b \) are not both equal to zero.

---

**INVESTIGATE & INQUIRE**

1. Copy and complete the table by expanding and simplifying each equation. Write each result in the form \( ax^2 + by^2 + 2gx + 2fy + c = 0 \).

<table>
<thead>
<tr>
<th>Standard Form</th>
<th>( ax^2 + by^2 + 2gx + 2fy + c = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) ( \frac{(x - 3)^2}{4} + \frac{(y + 2)^2}{25} = 1 )</td>
<td></td>
</tr>
<tr>
<td>b) ( (x - 3)^2 + (y - 1)^2 = 9 )</td>
<td></td>
</tr>
<tr>
<td>c) ( y + 3 = 2(x - 1)^2 )</td>
<td></td>
</tr>
<tr>
<td>d) ( \frac{(x - 1)^2}{9} - \frac{(y + 1)^2}{16} = 1 )</td>
<td></td>
</tr>
</tbody>
</table>
2. Identify each equation in standard form in question 1 as the equation of a circle, ellipse, hyperbola, or parabola.

3. a) For the circle written in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$, what do you notice about the values of $a$ and $b$?
   b) For a circle, if $f = 0$ and $g = 0$, what are the coordinates of the centre?

4. a) For the ellipse written in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$, what do you notice about the signs of $a$ and $b$?
   b) For an ellipse, if $f = 0$ and $g = 0$, what are the coordinates of the centre?

5. a) For the parabola written in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$, what do you notice about the values of $a$ and $b$?
   b) What is always true for the value of $a$ or $b$ for a parabola?

6. a) For the hyperbola written in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$, what do you notice about the signs of $a$ and $b$?
   b) For a hyperbola, if $f = 0$ and $g = 0$, what are the coordinates of centre?

7. If you are given the equation of a conic in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$, how can you identify the type of conic, without rewriting the equation in standard form?

8. Without rewriting in standard form, identify each of the following conics.
   a) $3x^2 + 24x - y + 50 = 0$
   b) $16x^2 - y^2 - 32x - 10y - 25 = 0$
   c) $4x^2 + y^2 + 8x - 4y + 4 = 0$
   d) $x^2 + y^2 - 6x - 2y + 1 = 0$

The type of conic represented by an equation in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$ can be identified using the signs and values of $a$ and $b$.

- For a circle, $a = b$.
- For an ellipse, $a$ and $b$ have the same sign, and $a \neq b$.
- For a parabola, either $a = 0$ or $b = 0$.
- For a hyperbola, $a$ and $b$ have opposite signs.

**Example 1** Sketching the Graph of a Conic

a) Identify the type of conic whose equation is $4x^2 + 9y^2 - 16x + 18y - 11 = 0$.
b) Write the equation in standard form.
c) Determine the key features and sketch the graph.
**Solution**

a) Since $a = 4$ and $b = 9$, $a$ and $b$ have the same sign, with $a \neq b$. The conic is an ellipse.

b) To write the equation in standard form, complete the square for both variables.

\[
4x^2 + 9y^2 - 16x + 18y - 11 = 0
\]

Add 11 to both sides:

\[
4x^2 + 9y^2 - 16x + 18y = 11
\]

Group the $x$ and $y$ terms:

\[
4(x^2 - 16x) + 9(y^2 + 18y) = 11
\]

Remove common factors:

\[
4(x^2 - 16x + 4 - 4) + 9(y^2 + 18y + 9 - 9) = 11
\]

Complete the square:

\[
4((x - 2)^2 - 16 + 4) + 9((y + 1)^2 - 9 + 9) = 11
\]

\[
4(x - 2)^2 + 9(y + 1)^2 = 36
\]

Divide both sides by 36:

\[
\frac{(x - 2)^2}{9} + \frac{(y + 1)^2}{4} = 1
\]

The equation in standard form is

\[
\frac{(x - 2)^2}{3^2} + \frac{(y + 1)^2}{2^2} = 1
\]

The ellipse is centred at $(h, k)$, or $(2, -1)$, and the major axis is parallel to the $x$-axis.

$a^2 = 9$, so $a = 3$

$b^2 = 4$, so $b = 2$

The major axis, which is parallel to the $x$-axis, has a length of $2a$, or 6.

The minor axis, which is parallel to the $y$-axis, has a length of $2b$, or 4.

The vertices are $V_1(h - a, k)$ and $V_2(h + a, k)$.

Substitute the values of $h$, $k$, and $a$.

The vertices are $V_1(2 - 3, -1)$ and $V_2(2 + 3, -1)$, or $V_1(-1, -1)$ and $V_2(5, -1)$.

The co-vertices are $(h, k - b)$ and $(h, k + b)$.

Substitute the values of $h$, $k$, and $b$.

The co-vertices are $(2, -1 - 2)$ and $(2, -1 + 2)$, or $(2, -3)$ and $(2, 1)$.
The foci are \( F_1(h - c, k) \) and \( F_2(h + c, k) \).

To find \( c \), we use \( a^2 = b^2 + c^2 \), with \( a = 3 \) and \( b = 2 \).

\[
a^2 = b^2 + c^2 = 3^2 = 2^2 + c^2 = 9 = 4 + c^2 = 5 = c
\]

The coordinates of the foci are \((2 - \sqrt{5}, -1)\) and \((2 + \sqrt{5}, -1)\), or approximately \((-0.24, -1)\) and \((4.24, -1)\).

Plot the vertices and co-vertices.

Draw a smooth curve through the points.

Label the foci and the graph.

**Example 2** Sketching the Graph of a Conic

a) Identify the type of conic whose equation is \( y^2 + 8x + 2y - 15 = 0 \).

b) Write the equation in standard form.

c) Determine the key features and sketch the graph.

**Solution**

a) Since \( a = 0 \) and \( b \neq 0 \), the conic is a parabola.

b) To write the equation in standard form, complete the square for the \( y \)-variable.

\[
y^2 + 8x + 2y - 15 = 0
\]

Add 15 to both sides:

\[
y^2 + 8x + 2y = 15
\]

Group the \( y \) terms:

\[
y^2 + 2y + 8x = 15
\]

Complete the square:

\[
y^2 + 2y + 1 - 1 + 8x = 15
\]

\[
(y + 1)^2 - 1 + 8x = 15
\]

\[
(y + 1)^2 + 8x = 16
\]

Rearrange:

\[
8x - 16 = -(y + 1)^2
\]

Remove a common factor:

\[
8(x - 2) = -(y + 1)^2
\]

Divide both sides by 8:

\[
x - 2 = -\frac{1}{8}(y + 1)^2
\]

The equation in standard form is \( x - 2 = -\frac{1}{8}(y + 1)^2 \).
c) The equation is in the form \( x - h = \frac{1}{4p}(y - k)^2 \).
The vertex is \( V(h, k) \).
\( h = 2, k = -1 \), so the vertex is \( V(2, -1) \).

Find the value of \( p \).
From the equation, \( \frac{1}{4p} = \frac{-1}{8} \)
\[ 4p = 4(-2) \]
\[ p = -2 \]

\( p < 0 \), so the parabola opens left. 
The focus \( F(h + p, k) \) is \( (2 + (-2), -1) \) or \( (0, -1) \).
The directrix is \( x = h - p \)
\[ x = 2 - (-2) \]
\[ x = 4 \]

The axis of symmetry is \( y = k \) or \( y = -1 \).
Sketch and label the graph.

---

**Example 3**  Shock Wave

A shock wave from an aircraft that breaks the sound barrier intersects the ground in a curve with the equation \( x^2 - 4y^2 + 4x + 24y - 36 = 0 \).

a) Identify the type of conic.
b) Write the equation in standard form.
c) Determine the key features and sketch the graph.

**Solution**

a) Since \( a \) and \( b \) have opposite signs, the conic is a hyperbola.
b) To write the equation in standard form, complete the square for both variables.

\[ x^2 - 4y^2 + 4x + 24y - 36 = 0 \]

Add 36 to both sides:

\[ x^2 - 4y^2 + 4x + 24y = 36 \]

Group the x and y terms:

\[ x^2 + 4x - 4y^2 + 24y = 36 \]

Remove a common factor:

\[ x^2 + 4x - 4(y^2 - 6y) = 36 \]

Complete the square:

\[ x^2 + 4x + 4 - 4(y^2 - 6y + 9) + 36 = 36 \]

\[ (x + 2)^2 - 4(y - 3)^2 = 4 \]

Divide both sides by 4:

\[ \frac{(x + 2)^2}{4} - \frac{(y - 3)^2}{1} = 1 \]

The equation in standard form is \( \frac{(x + 2)^2}{4} - (y - 3)^2 = 1 \).

c) The equation is in the form \( \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \).

The centre is \( C(h, k) = (-2, 3) \).

The transverse axis is parallel to the x-axis.

\[ a^2 = 4, \text{ so } a = 2 \]

\[ b^2 = 1, \text{ so } b = 1 \]

The vertices are \( V_1(h - a, k) \) and \( V_2(h + a, k) \).

Substitute the values for \( h, k, \) and \( a \).

\( V_1(-2 - 2, 3) \) and \( V_2(-2 + 2, 3) \) or \( V_1(-4, 3) \) and \( V_2(0, 3) \).

The co-vertices are \( (h, k - b) \) and \( (h, k + b) \).

Substitute the values for \( h, k, \) and \( b \).

The coordinates of the co-vertices are \((-2, 3 - 1)\) and \((-2, 3 + 1)\) or \((-2, 2)\) and \((-2, 4)\).

The length of the transverse axis is

\[ 2a = 2(2) \]

\[ = 4 \]

The length of the conjugate axis is

\[ 2b = 2(1) \]

\[ = 2 \]
The coordinates of the foci are \( F_1(h - c, k) \) and \( F_2(h + c, k) \).

To find \( c \), use \( c^2 = a^2 + b^2 \), with \( a = 2 \) and \( b = 1 \).

\[
c^2 = 2^2 + 1^2 = 5
\]

\[c = \sqrt{5}\]

The coordinates of the foci are \((-2 - \sqrt{5}, 3)\) and \((-2 + \sqrt{5}, 3)\), or approximately \((-4.24, 3)\) and \((0.24, 3)\).

Sketch and label the graph.

**Key Concepts**

- The equation of a conic can be written in the form \( ax^2 + by^2 + 2gx + 2fy + c = 0 \), where \( a \) and \( b \) are not both equal to zero.
- The type of conic represented by an equation in the form \( ax^2 + by^2 + 2gx + 2fy + c = 0 \) can be identified using the signs and values of \( a \) and \( b \).
  * For a circle, \( a = b \).
  * For an ellipse, \( a \) and \( b \) have the same sign, and \( a \neq b \).
  * For a parabola, \( a = 0 \) or \( b = 0 \).
  * For a hyperbola, \( a \) and \( b \) have opposite signs.
- To graph a conic section whose equation is in the form \( ax^2 + by^2 + 2gx + 2fy + c = 0 \), first use the method of completing the square to write the equation in standard form.

**Communicate Your Understanding**

1. Identify each of the following conics.
   a) \( x^2 + 4y^2 - 16 = 0 \)  
   b) \( 2x^2 - 2x - 6y - 3 = 0 \)  
   c) \( x^2 + y^2 - 4x + 8y - 44 = 0 \)  
   d) \( x^2 - 9y^2 - 14x + 36y + 4 = 0 \)
2. Describe how you would write an equation in standard form for the conic defined by \( 4x^2 + 25y^2 - 16x + 50y - 9 = 0 \).
Practise

A

1. Identify the type of conic by inspection.
   a) \(x^2 - 2y^2 - 6x + 4y - 2 = 0\)
   b) \(2x^2 + y^2 - 6x - 4y - 3 = 0\)
   c) \(x^2 + y^2 - 5x + 4y + 3 = 0\)
   d) \(3y^2 + 6x - 6y - 9 = 0\)
   e) \(2x^2 - 3y^2 - 6x - 1 = 0\)
   f) \(3x^2 - 4y^2 + 3x + 6y - 1 = 0\)
   g) \(2x^2 - 6x + 9y = 0\)

2. For each of the following equations,
   i) identify the type of conic
   ii) write the equation in standard form
   iii) determine the key features and sketch the graph
   a) \(x^2 + y^2 - 2x - 6y - 15 = 0\)
   b) \(4x^2 + y^2 + 24x - 4y - 24 = 0\)
   c) \(x^2 + 6x - 8y + 25 = 0\)
   d) \(y^2 - 4y - 8x + 12 = 0\)
   e) \(x^2 + 16y^2 + 8x - 96y + 144 = 0\)
   f) \(x^2 + y^2 + 4x - 6y - 23 = 0\)
   g) \(2x^2 - 2y^2 + 4x - 4y + 1 = 0\)
   h) \(y^2 - 4y + 4x + 8 = 0\)
   i) \(x^2 - 2y^2 - 6x - 4y - 2 = 0\)

3. For each conic, write an equation in standard form and in the form
   \(ax^2 + by^2 + 2gx + 2fy + c = 0\).
   a) [Graph]

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Apply, Solve, Communicate

4. Inquiry/Problem Solving a) For the equation $x^2 + by^2 - 4 = 0$, determine the value(s) of $b$ that result in an equation of
   i) a circle     ii) a parabola
   iii) an ellipse  iv) a hyperbola
b) Give an example to illustrate each answer in part a).

5. a) For the equation $ax^2 - y^2 + 9 = 0$, determine the value(s) of $a$ that will result in an equation of
   i) a circle     ii) a parabola
   iii) an ellipse  iv) a hyperbola
b) Give an example to illustrate each answer in part a).

6. Inquiry/Problem Solving The three squares in the diagram are centred at the same point. The red border has the same area as the smallest square.
   a) How are $p$ and $q$ related?
   b) Graph the relation.

7. Application When a plane breaks the sound barrier, a shock wave in the shape of a cone is produced. If the plane is flying parallel to the ground, the shock wave intersects the ground in a branch of a hyperbola.
   a) Use your knowledge of the intersection of a plane and a cone to explain why the intersection is a branch of a hyperbola.
   b) If the intersection of a shock wave with the ground can be modelled by the equation $x^2 + 25y^2 - 8x + 100y + 91 = 0$, describe how the plane is flying.
   c) Is it possible for the intersection of a shock wave with the ground to be modelled by the equation $4x^2 + 4y^2 + 36y + 5 = 0$? Explain.
8. **Degenerate conics** The rules for identifying a type of conic from its equation do not always apply. For example, in \( x^2 + y^2 + 2x - 4y + 5 = 0 \), \( a = b \), so the equation appears to model a circle.

Completing the square gives

\[
\begin{aligned}
x^2 + y^2 + 2x - 4y + 5 &= 0 \\
x^2 + 2x + 1 - 1 + y^2 - 4y + 4 - 4 + 5 &= 0 \\
(x + 1)^2 + (y - 2)^2 &= 0
\end{aligned}
\]

The ordered pair \((-1, 2)\) satisfies the equation. The solution is \((-1, 2)\).

Therefore, the graph of \( x^2 + y^2 + 2x - 4y + 5 = 0 \) is a point, not a circle. Because the equation appears to model a circle, the graph is referred to as a degenerate circle.

Changing the equation to \( x^2 + y^2 + 2x - 4y + 6 = 0 \) and completing the square gives \((x + 1)^2 + (y - 2)^2 = -1\).

This equation has no solution, since the sum of two squares cannot be negative. So, the equation \( x^2 + y^2 + 2x - 4y + 6 = 0 \) is degenerate.

i) Identify the type of conic that each of the following equations appears to model.

ii) Verify that each equation is degenerate.

iii) Graph the equation, if possible.

a) \( 4x^2 + y^2 - 8x + 2y + 6 = 0 \)

b) \( x^2 + y^2 - 2x - 6y + 10 = 0 \)

c) \( 3x^2 + 4y^2 - 6x - 24y + 39 = 0 \)

d) \( 9x^2 - y^2 + 18x + 6y = 0 \)

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**LOGIC Power**

What are the four moves that X should not play, if X wants to stop O from winning?