2.1 The Complex Number System

The approximate speed of a car prior to an accident can be found using the length of the tire marks left by the car after the brakes have been applied. The formula \( s = \frac{\sqrt{121d}}{100} \) gives the speed, \( s \), in kilometres per hour, where \( d \) is the length of the tire marks, in metres. Radical expressions like \( \sqrt{121d} \) can be simplified.

**INVESTIGATE & INQUIRE**

Copy the table. Complete it by replacing each \( \square \) with a whole number.

| \( \sqrt{4 \times 9} = \square \) | \( \sqrt{4 \times 9} = \square \times \square = \square \) |
| \( \sqrt{9 \times 16} = \square \) | \( \sqrt{9 \times 16} = \square \times \square = \square \) |
| \( \sqrt{25 \times 4} = \square \) | \( \sqrt{25 \times 4} = \square \times \square = \square \) |
| \( \frac{36}{4} = \sqrt{\square} = \square \) | \( \sqrt{\frac{36}{4}} = \square = \square \) |
| \( \sqrt{\frac{100}{25}} = \sqrt{\square} = \square \) | \( \sqrt{\frac{100}{25}} = \frac{\sqrt{\square}}{\sqrt{\square}} = \frac{\square}{\square} = \square \) |
| \( \sqrt{\frac{144}{9}} = \sqrt{\square} = \square \) | \( \sqrt{\frac{144}{9}} = \frac{\sqrt{\square}}{\sqrt{\square}} = \frac{\square}{\square} = \square \) |

1. Compare the two results in each of the first three rows of the table.

2. If \( a \) and \( b \) are whole numbers, describe how \( \sqrt{ab} \) is related to \( \sqrt{a} \times \sqrt{b} \).

3. **Technology** Use a calculator to test your statement from question 2 for each of the following.
   a) \( \sqrt{5 \times 9} \)  
   b) \( \sqrt{3 \times 7} \)

4. Compare the two results in each of the last three rows of the table.

5. If \( a \) and \( b \) are whole numbers, and \( b \neq 0 \), describe how \( \sqrt{\frac{a}{b}} \) is related to \( \frac{\sqrt{a}}{\sqrt{b}} \).
6. **Technology** Use a calculator to test your statement from question 5 for each of the following.
   a) \(36 \div \sqrt{2}\)  
   b) \(\sqrt{15} \div \sqrt{3}\)

7. Write the expression \(\sqrt{121d}\) in the form \(\sqrt{d}\), where \(d\) represents a whole number.

8. Determine the speed of a car that leaves tire marks of each of the following lengths. Round the speed to the nearest kilometre per hour, if necessary.
   a) 64 m  
   b) 100 m  
   c) 15 m

The following properties are used to simplify radicals.

- \(\sqrt{ab} = \sqrt{a} \times \sqrt{b}\), \(a \geq 0, \ b \geq 0\)
- \(\sqrt[\frac{a}{b}] = \frac{\sqrt{a}}{\sqrt{b}}\), \(a \geq 0, \ b > 0\)

A radical is in simplest form when

- the radicand has no perfect square factors other than 1
- the radicand does not contain a fraction
- no radical appears in the denominator of a fraction

**Example 1 Simplifying Radicals**

Simplify.

a) \(\sqrt{75}\)  
   b) \(\frac{\sqrt{48}}{\sqrt{6}}\)  
   c) \(\sqrt\frac{2}{9}\)

**Solution**

a) \(\sqrt{75} = \sqrt{25} \times \sqrt{3}\)
   \(= 5\sqrt{3}\)

b) \(\frac{\sqrt{48}}{\sqrt{6}} = \sqrt{\frac{48}{6}}\)
   \(= \sqrt{8}\)
   \(= \sqrt{4} \times \sqrt{2}\)
   \(= 2\sqrt{2}\)

Recall that the radicand is the expression under the radical sign.
c) \[ \frac{\sqrt{2}}{9} = \frac{\sqrt{2}}{9} \]
\[ = \frac{\sqrt{2}}{3} \text{ or } \frac{1}{3} \sqrt{2} \]

Note that, in Example 1, numbers like \(\sqrt{75}\) and \(\frac{\sqrt{2}}{9}\) are called **entire radicals**.

Numbers like \(5\sqrt{3}\) and \(\frac{1}{3} \sqrt{2}\) are called **mixed radicals**.

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**Example 2 Multiplying Radicals**

Simplify.

a) \(9\sqrt{2} \times 4\sqrt{7}\)  

b) \(2\sqrt{3} \times 5\sqrt{6}\)

**Solution**

a) \(9\sqrt{2} \times 4\sqrt{7} = 9 \times 4 \times \sqrt{2} \times \sqrt{7} = 36\sqrt{14}\)

b) \(2\sqrt{3} \times 5\sqrt{6} = 2 \times 5 \times \sqrt{3} \times \sqrt{6} = 10\sqrt{18} = 10 \times \sqrt{9 \times \sqrt{2}} = 10 \times 3 \sqrt{2} = 30\sqrt{2}\)

**Example 3 Simplifying Radical Expressions**

Simplify \(\frac{6 - \sqrt{45}}{3}\).

**Solution**

\[\frac{6 - \sqrt{45}}{3} = \frac{6 - \sqrt{9 \times 5}}{3} = \frac{6 - 3\sqrt{5}}{3} = 2 - \sqrt{5}\]
In mathematics, we can find the square roots of negative numbers as well as positive numbers. Mathematicians have invented a number defined as the principal square root of negative one. This number, \( i \), is the **imaginary unit**, with the following properties.

\[
i = \sqrt{-1} \quad \text{and} \quad i^2 = -1
\]

In general, if \( x \) is a positive real number, then \( \sqrt{-x} \) is a **pure imaginary number**, which can be defined as follows.

\[
\sqrt{-x} = \sqrt{-1} \times \sqrt{x} = i \sqrt{x}
\]

So, \( \sqrt{-5} = \sqrt{-1} \times 5 = i \times 5 \). To ensure that \( \sqrt{-xi} \) is not read as \( \sqrt{\sqrt{x}} \), we write \( \sqrt{-xi} \) as \( i \sqrt{x} \).

Despite their name, pure imaginary numbers are just as real as real numbers. When the radicand is a negative number, there is an extra rule for expressing a radical in simplest form.

- A radical is in simplest form when the radicand is positive.

Numbers such as \( i \), \( i \sqrt{6} \), \( 2i \), and \(-3i \) are examples of pure imaginary numbers in simplest form.

**Example 4 Simplifying Pure Imaginary Numbers**

Simplify.

a) \( \sqrt{-25} \)  \quad b) \( \sqrt{-12} \)

**Solution**

a) \( \sqrt{-25} = \sqrt{-1} \times \sqrt{25} = i \times 5 = 5i \)

b) \( \sqrt{-12} = \sqrt{-1} \times \sqrt{12} = i \times \sqrt{4} \times \sqrt{3} = i \times 2 \times \sqrt{3} = 2i \sqrt{3} \)
When two pure imaginary numbers are multiplied, the result is a real number.

**Example 5 Multiplying Pure Imaginary Numbers**

Evaluate.

a) \(3i \times 4i\)  
\[= 12 \times (-1) = -12\]

b) \(2i \times (-5i)\)  
\[= -10 \times (-1) = 10\]

c) \((3i\sqrt{2})^2\)  
\[= 9 \times (-1) \times 2 = -18\]

A **complex number** is a number in the form \(a + bi\), where \(a\) and \(b\) are real numbers and \(i\) is the imaginary unit. We call \(a\) the **real part** and \(bi\) the **imaginary part** of a complex number. Examples of complex numbers include \(5 + 2i\) and \(4 - 3i\). Complex numbers are used for applications of mathematics in engineering, physics, electronics, and many other areas of science.

If \(b = 0\), then \(a + bi = a\). So, a real number, such as 5, can be thought of as a complex number, since 5 can be written as \(5 + 0i\).

If \(a = 0\), then \(a + bi = bi\). Numbers of the form \(bi\), such as \(7i\), are **pure imaginary numbers**. Complex numbers in which neither \(a = 0\) nor \(b = 0\) are referred to as **imaginary numbers**.
The following diagram summarizes the complex number system.

**Complex Numbers**

\[ a + bi, \text{ where } a \text{ and } b \text{ are real numbers and } i = \sqrt{-1}. \]

**Real Numbers**

\( \left( \text{e.g., } 5, \sqrt{2}, -7, 3.6, \frac{-2}{3} \right) \)

**Imaginary Numbers**

\( \left( \text{e.g., } 4 + 3i, 3 - 2i \right) \)

**Rational Numbers**

Can be expressed as the ratio of two integers.

\( \left( \text{e.g., } 3, -\sqrt{4}, 0.27, -\frac{6}{7} \right) \)

**Irrational Numbers**

Cannot be expressed as the ratio of two integers.

\( \left( \text{e.g., } \sqrt{7}, -\sqrt{2}, \pi \right) \)

**Integers**

\( \text{Whole numbers and their opposites} \)

\( \left( \text{e.g., } 4, -4, 0, 9, -9 \right) \)

**Whole Numbers**

Positive integers and zero

\( \left( \text{e.g., } 0, 3, 7, 11 \right) \)

**Natural Numbers**

Positive integers

\( \left( \text{e.g., } 1, 5, 8, 23 \right) \)

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**Example 6** Simplifying Complex Numbers

Simplify.

a) \( 3 - \sqrt{-24} \)

b) \( \frac{10 + \sqrt{-32}}{2} \)

**Solution**

a) \( 3 - \sqrt{-24} = 3 - \sqrt{-1} \times \sqrt{24} \)

\[ = 3 - i \times 4 \times \sqrt{6} \]

\[ = 3 - 2i \sqrt{6} \]

The expression \( 3 - 2i \sqrt{6} \) cannot be simplified further, because the real part and the imaginary part are unlike terms.
**Key Concepts**

- Radicals are simplified using the following properties.
  \[ \sqrt{ab} = \sqrt{a} \times \sqrt{b}, \ a \geq 0, \ b \geq 0 \]
  \[ \sqrt[2]{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, \ a \geq 0, \ b > 0 \]
- The number \( i \) is the imaginary unit, where \( i^2 = -1 \) and \( i = \sqrt{-1} \).
- A complex number is a number in the form \( a + bi \), where \( a \) and \( b \) are real numbers and \( i \) is the imaginary unit.

**Communicate Your Understanding**

1. Describe the difference between an entire radical and a mixed radical.
2. Describe how you would simplify
   a) \( \sqrt{60} \)  
   b) \( \frac{\sqrt{14}}{2} \)  
   c) \( \frac{10 + \sqrt{20}}{2} \)
3. Describe how you would simplify
   a) \( \sqrt{-28} \)  
   b) \( \frac{9 + \sqrt{-54}}{3} \)
4. Describe how you would evaluate \((-3i)^2\).

**Practice**

**A**

1. Simplify.
   a) \( \sqrt{12} \)  
   b) \( \sqrt{20} \)  
   c) \( \sqrt{45} \)  
   d) \( \sqrt{50} \)  
   e) \( \sqrt{24} \)  
   f) \( \sqrt{63} \)  
   g) \( \sqrt{200} \)  
   h) \( \sqrt{32} \)  
   i) \( \sqrt{44} \)  
   j) \( \sqrt{60} \)  
   k) \( \sqrt{18} \)  
   l) \( \sqrt{54} \)  
   m) \( \sqrt{128} \)  
   n) \( \sqrt{90} \)  
   o) \( \sqrt{125} \)
2. Simplify.
   a) $\sqrt{\frac{14}{7}}$  b) $\sqrt{\frac{10}{2}}$  c) $\sqrt{\frac{60}{3}}$  d) $\sqrt{\frac{40}{5}}$
   e) $\sqrt{\frac{33}{3}}$  f) $\sqrt{\frac{7}{4}}$  g) $\sqrt{\frac{20}{9}}$  h) $\frac{3}{2}$
   i) $\sqrt{\frac{27\times15}{3\times5}}$  j) $\sqrt{\frac{12\times75}{4\times3}}$  k) $\sqrt{\frac{4\times2}{8}}$  l) $\sqrt{\frac{2\times2}{18}}$

   a) $\sqrt{2 \times \sqrt{10}}$  b) $\sqrt{3 \times \sqrt{6}}$
   c) $\sqrt{15 \times \sqrt{5}}$  d) $\sqrt{7 \times \sqrt{11}}$
   e) $\sqrt{4 \times 3 \times \sqrt{7}}$  f) $\sqrt{3 \times 6 \times 3 \times \sqrt{6}}$
   g) $\sqrt{2 \times 2 \times 3 \times \sqrt{5}}$  h) $\sqrt{2 \times 5 \times 3 \times 10}$
   i) $\sqrt{3 \times 3 \times 4 \times \sqrt{15}}$  j) $\sqrt{4 \times 7 \times 2 \times 14}$
   k) $\sqrt{6 \times 3 \times \sqrt{2}}$  l) $\sqrt{2 \times 7 \times 3 \times 1 \times \sqrt{7}}$

4. Simplify.
   a) $\frac{10 + 15 \sqrt{5}}{5}$  b) $\frac{21 - \sqrt{7} \times 6}{6 + \sqrt{8}}$  c) $\frac{6 + \sqrt{8}}{2}$
   d) $\frac{12 - \sqrt{27}}{3}$  e) $\frac{-10 - \sqrt{50}}{5}$  f) $\frac{-12 + \sqrt{48}}{4}$

5. Simplify.
   a) $\sqrt{-9}$  b) $\sqrt{-25}$  c) $\sqrt{-81}$  d) $\sqrt{-5}$
   e) $\sqrt{-13}$  f) $\sqrt{-23}$  g) $\sqrt{-12}$  h) $\sqrt{-40}$
   i) $\sqrt{-54}$  j) $\sqrt{-4}$  k) $\sqrt{-20}$  l) $\sqrt{-60}$

6. Evaluate.
   a) $5i \times 5i$  b) $2i \times 3i$
   c) $(-2i) \times (-2i)$  d) $(-3i) \times (-4i)$
   e) $2i \times (-5i)$  f) $(-3i) \times 6i$

7. Simplify.
   a) $i^3$  b) $i^4$
   c) $i^5$  d) $4i \times 5i$
   e) $5i^2$  f) $-i^7$
   g) $3(-2i)^2$  h) $i(4i)^3$
   i) $(3i)(-6i)$  j) $(i/2)^2$
   k) $-\left(\sqrt{5}\right)^2$  l) $(i/6)(-i/6)$
   m) $(2i/3)^2$  n) $\left(-5i/2\right)^2$
   o) $\left(4i/5\right)(-2i/5)$

8. Simplify.
   a) $4 + \sqrt{-20}$  b) $7 - \sqrt{-18}$
   c) $10 + \sqrt{-75}$  d) $11 - \sqrt{-63}$
   e) $-2 - \sqrt{-90}$  f) $-6 - \sqrt{-52}$

   a) $\frac{15 + 20i\sqrt{5}}{5}$  b) $\frac{14 - 28i\sqrt{6}}{14}$
   c) $\frac{10 - \sqrt{-16}}{2}$  d) $\frac{12 + \sqrt{-27}}{3}$
   e) $\frac{-8 + \sqrt{-32}}{4}$  f) $\frac{-21 - \sqrt{-98}}{7}$

Apply, Solve, Communicate

10. Express each of the following as an integer.
    a) $\sqrt{5^2}$  b) $\left(\sqrt{5}\right)^2$  c) $\sqrt{(-5)^2}$  d) $-\sqrt{5^2}$
    e) $-\left(\sqrt{5}\right)^2$  f) $-\sqrt{(-5)^2}$

11. Communication  Classify each of the following numbers into one or more of these sets: real, rational, irrational, complex, imaginary, pure imaginary. Explain your reasoning.
    a) $\sqrt{5}$  b) $\sqrt{-4}$  c) $1 + \sqrt{3}$  d) $3 + i\sqrt{6}$
12. **Measurement** Express the exact area of the triangle in simplest radical form.

13. **Measurement** A square has an area of 675 cm\(^2\). Express the side length in simplest radical form.

14. **Application** There are many variations on the game of chess. Most are played on square boards that consist of a number of small squares. However, some variations do not use the familiar 64-square board.
   
   a) If each small square on a Grand Chess board is 2 cm by 2 cm, each diagonal of the whole board measures \(\sqrt{800}\) cm. How many small squares are on the board?
   
   b) A Japanese variation of chess is called Chu Shogi. If each small square on a Chu Shogi board measures 3 cm by 3 cm, each diagonal of the whole board measures \(\sqrt{2592}\) cm. How many small squares are on the board?

15. **Pattern**
   
   a) Simplify \(i^2, i^3, i^4, i^5, i^6, i^7, i^8, i^9, i^{10}, i^{11}\), and \(i^{12}\).
   
   b) Describe the pattern in the values.
   
   c) Describe how to simplify \(i^n\), where \(n\) is a whole number.
   
   d) Simplify \(i^{48}, i^{94}, i^{85}\), and \(i^{99}\).

16. **Measurements of lengths and areas**
   
   a) Determine the length of the diagonal of each of the following Ontario flags. Write each answer in simplest radical form.

   b) Describe the relationship between the length of the diagonal and either dimension of the flag.
   
   c) Use the relationship from part b) to predict the length of the diagonal of a 150 cm by 75 cm Ontario flag. Leave your answer in simplest radical form.
d) Describe the relationship between the length of the diagonal and the area of the Ontario flag.
e) Use the relationship from part d) to predict the length of the diagonal of an Ontario flag with an area of 24,200 cm². Leave your answer in simplest radical form.

17. **Communication** Check if \( x = -i \sqrt{3} \) is a solution to the equation \( x^2 + 3 = 0 \). Justify your reasoning.

18. Simplify each of the following by first expressing it as the product of two cube roots.
   a) \( \sqrt[3]{16} \)
   b) \( \sqrt[3]{32} \)
   c) \( \sqrt[3]{54} \)
   d) \( \sqrt[3]{81} \)

19. **Equations** Solve. Express each answer in simplest radical form.
   a) \( \sqrt{2} = \sqrt{14} \)
   b) \( 5x = \sqrt{50} \)
   c) \( \sqrt{\frac{x}{3}} = \sqrt{6} \)
   d) \( \sqrt{\frac{30}{x}} = \sqrt{5} \)

20. a) For the property \( \sqrt[3]{a} \times \sqrt[3]{b} = \sqrt[3]{ab} \), explain the restrictions \( a \geq 0, b \geq 0 \).
b) For the property \( \sqrt[3]{a} \div \sqrt[3]{b} = \sqrt[3]{\frac{a}{b}} \), the restrictions are \( a \geq 0, b > 0 \).
   Why is the second restriction not \( b \geq 0 \)?

**Pattern: Power**

Subtracting 9 from the two-digit positive integer 21 results in the reversal of the digits to give 12.

1. List all two-digit positive integers for which the digits are reversed when you subtract 9.
2. Find all the two-digit positive integers for which the digits are reversed when you subtract
   a) 18   b) 27   c) 36
3. Describe the pattern in words.
4. Use the pattern to find all the two-digit positive integers for which the digits are reversed when you subtract
   a) 54   b) 72