3.3 Horizontal and Vertical Translations of Functions

When an object is dropped from the top of a bridge over a body of water, the approximate height of the falling object above the water is given by the function

\[ h(t) = -5t^2 + d \]

where \( h(t) \) metres is the height of the object \( t \) seconds after it is dropped, and \( d \) metres is the height of the bridge.

**INVESTIGATE & INQUIRE**

1. The table includes the approximate heights, in metres, of three famous Canadian bridges. Write the function that describes the height of a falling object above the water \( t \) seconds after it is dropped from the top of each bridge.

<table>
<thead>
<tr>
<th>Bridge</th>
<th>Height (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ambassador Bridge</td>
<td>45</td>
</tr>
<tr>
<td>Confederation Bridge</td>
<td>60</td>
</tr>
<tr>
<td>Capilano Canyon Suspension Bridge</td>
<td>70</td>
</tr>
</tbody>
</table>

2. Graph \( h(t) \) versus \( t \) for the three functions on the same set of axes or in the same viewing window of a graphing calculator.


4. For a given \( t \)-coordinate, how does the \( h \)-coordinate of a point on the graph for the Confederation Bridge compare with the \( h \)-coordinate of a point on the graph for the Ambassador Bridge? Explain why.

5. For a given \( t \)-coordinate, how does the \( h \)-coordinate of a point on the graph for the Capilano Bridge compare with the \( h \)-coordinate of a point on the graph for the Ambassador Bridge? Explain why.

6. Graph the three functions \( y = -5x^2 + c \) for \( c = 45, c = 60, \) and \( c = 70 \) on the same set of axes or in the same viewing window of a graphing calculator. If the domain of the three functions is the set of real numbers, how do the three graphs compare with the three graphs of \( h(t) \) versus \( t \) from question 2? Explain why.

7. How do the three graphs from question 6 compare with the graph of \( y = x^2 \) over the same domain? Explain why.
**Example 1  Positive Vertical Translation**

a) Graph the functions \( y = x^2 \) and \( y = x^2 + 2 \) on the same set of axes.

b) How does the graph of \( y = x^2 \) compare to the graph of \( y = x^2 + 2 \)?

**Solution**

a) \( y = x^2 \)  
\[
\begin{array}{c|c}
 x & y \\
\hline
 3 & 9 \\
 2 & 4 \\
 1 & 1 \\
 0 & 0 \\
-1 & 1 \\
-2 & 4 \\
-3 & 9 \\
\end{array}
\]

\[
\begin{array}{c|c}
 x & y \\
\hline
 3 & 11 \\
 2 & 6 \\
 1 & 3 \\
 0 & 2 \\
-1 & 3 \\
-2 & 6 \\
-3 & 11 \\
\end{array}
\]

The graphs of \( y = x^2 \) and \( y = x^2 + 2 \) are congruent.

The graph of the function \( y = x^2 + 2 \) is obtained when the graph of the function \( y = x^2 \) undergoes a vertical translation of 2 units in the positive direction, that is, upward.

The graph of the function \( y = x^2 + 2 \) can also be obtained from the graph of \( y = x^2 \) by adding 2 to each \( y \)-value on the graph of \( y = x^2 \). The point \((x, y)\) on the graph of \( y = x^2 \) is transformed to become the point \((x, y + 2)\) on the graph of \( y = x^2 + 2 \).

The graphs show that the functions have the same domain but different ranges. The domain of each function is the set of real numbers. The range of \( y = x^2 \) is \( y \geq 0 \). The range of \( y = x^2 + 2 \) is \( y \geq 2 \).

**Example 2  Vertical Translations**

How do the graphs of \( y = \sqrt{x} + 3 \) and \( y = \sqrt{x} - 2 \) compare with the graph of \( y = \sqrt{x} \), where \( x \geq 0 \).

The window variables include \( X_{\text{min}} = -4 \), \( X_{\text{max}} = 4 \), \( Y_{\text{min}} = 0 \), and \( Y_{\text{max}} = 12 \).

Note that the transformations shown in this chapter can be performed using a graphing software program, such as Zap-a-Graph. For details of how to do this, refer to the Zap-a-Graph section of Appendix C.
**Solution**

\[ y = \sqrt{x} \quad y = \sqrt{x} + 3 \quad y = \sqrt{x} - 2 \]

\[
\begin{array}{c|c}
 x & y \\
0 & 0 \\
1 & 1 \\
4 & 2 \\
9 & 3 \\
16 & 4 \\
\end{array}
\quad \begin{array}{c|c}
 x & y \\
0 & 3 \\
1 & 4 \\
4 & 5 \\
9 & 6 \\
16 & 7 \\
\end{array}
\quad \begin{array}{c|c}
 x & y \\
0 & -2 \\
1 & -1 \\
4 & 0 \\
9 & 1 \\
16 & 2 \\
\end{array}
\]

The graph of the function \( y = \sqrt{x} \) is the graph of the function \( y = \sqrt{x} + 3 \) with a vertical translation of 3 units upward. The point \((x, y)\) on the graph of \( y = \sqrt{x} \) is transformed to become the point \((x, y + 3)\) on the graph of \( y = \sqrt{x} + 3 \).

Similarly, the graph of the function \( y = \sqrt{x} - 2 \) is the graph of the function \( y = \sqrt{x} \) with a vertical translation of 2 units downward. The point \((x, y)\) on the graph of \( y = \sqrt{x} \) is transformed to become the point \((x, y - 2)\) on the graph of \( y = \sqrt{x} - 2 \).

All three graphs are congruent and have domain \( x \geq 0 \). The range of \( y = \sqrt{x} \) is \( y \geq 0 \), of \( y = \sqrt{x} + 3 \) is \( y \geq 3 \), and of \( y = \sqrt{x} - 2 \) is \( y \geq -2 \).

The results from Examples 1 and 2 can be generalized for all functions as follows.

The graph of \( y = f(x) + k \) is congruent to the graph of \( y = f(x) \).

If \( k > 0 \), the graph of \( y = f(x) + k \) is the graph of \( y = f(x) \) translated upward by \( k \) units.

If \( k < 0 \), the graph of \( y = f(x) + k \) is the graph of \( y = f(x) \) translated downward by \( k \) units.
Example 3  Horizontal Translations

How do the graphs of $y = \sqrt{x + 2}$ and $y = \sqrt{x - 3}$ compare to the graph of $y = \sqrt{x}$.

Solution

Complete tables of values using convenient values for $x$, or use a graphing calculator.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x$</th>
<th>$y$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>14</td>
<td>5</td>
<td>19</td>
<td>4</td>
</tr>
</tbody>
</table>

The graphs of $y = \sqrt{x}$, $y = \sqrt{x + 2}$, and $y = \sqrt{x - 3}$ are congruent.

The graph of $y = \sqrt{x + 2}$ is obtained when the graph of $y = \sqrt{x}$ is translated horizontally 2 units to the left.

The graph of $y = \sqrt{x + 2}$ is also obtained from the graph of $y = \sqrt{x}$ by subtracting 2 from each $x$-value on the graph of $y = \sqrt{x}$.

The point $(x, y)$ on the graph of $y = \sqrt{x}$ is transformed to become the point $(x - 2, y)$ on the graph of $y = \sqrt{x + 2}$.

Similarly, the graph of $y = \sqrt{x - 3}$ is obtained when the graph of $y = \sqrt{x}$ is translated horizontally 3 units to the right.

The graph of $y = \sqrt{x - 3}$ is also obtained from the graph of $y = \sqrt{x}$ by adding 3 to each $x$-value on the graph of $y = \sqrt{x}$.

The point $(x, y)$ on the graph of $y = \sqrt{x}$ is transformed to become the point $(x + 3, y)$ on the graph of $y = \sqrt{x - 3}$.

All three graphs have the same range, $y \geq 0$.

The domain of $y = \sqrt{x}$ is $x \geq 0$, of $y = \sqrt{x + 2}$ is $x \geq -2$, and of $y = \sqrt{x - 3}$ is $x \geq 3$.
The results from Example 3 can be generalized for all functions as follows.

The graph of \( y = f(x - h) \) is congruent to the graph of \( y = f(x) \).

If \( h > 0 \), the graph of \( y = f(x - h) \) is the graph of \( y = f(x) \) translated to the right by \( h \) units.

If \( h < 0 \), the graph of \( y = f(x - h) \) is the graph of \( y = f(x) \) translated to the left by \( h \) units.

**Example 4  Horizontal and Vertical Translations**

Sketch the graph of \( y = (x - 3)^2 + 4 \).

**Solution**

Sketch the graph of \( y = x^2 \).

Translate the graph of \( y = x^2 \) three units to the right to obtain the graph of \( y = (x - 3)^2 \).

Translate the graph of \( y = (x - 3)^2 \) four units upward to obtain the graph of \( y = (x - 3)^2 + 4 \).

The point \((x, y)\) on the function \( y = x^2 \) is transformed to become the point \((x + 3, y + 4)\). For example, \((0, 0)\) becomes \((3, 4)\).

Note that, in Example 4, you could graph the three functions using a graphing calculator. However, it is not necessary to graph \( y = x^2 \) or \( y = (x - 3)^2 \) before graphing \( y = (x - 3)^2 + 4 \).

The window variables include \( X_{\text{min}} = -4, X_{\text{max}} = 7, Y_{\text{min}} = -2, \) and \( Y_{\text{max}} = 10 \).
Key Concepts

- A function and its translation image are congruent.
- The table summarizes translations of the function \( y = f(x) \).

<table>
<thead>
<tr>
<th>Translation</th>
<th>Mathematical Form</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical</td>
<td>( y = f(x) + k )</td>
<td>If ( k &gt; 0 ), then ( f ) units upward</td>
</tr>
<tr>
<td></td>
<td></td>
<td>If ( k &lt; 0 ), then ( f ) units downward</td>
</tr>
<tr>
<td>Horizontal</td>
<td>( y = f(x - h) )</td>
<td>If ( h &gt; 0 ), then ( h ) units to the right</td>
</tr>
<tr>
<td></td>
<td></td>
<td>If ( h &lt; 0 ), then ( h ) units to the left</td>
</tr>
</tbody>
</table>

Communicate Your Understanding

1. Starting with the graph of \( y = \sqrt{x} \), describe how you would sketch each of the following graphs.
   a) \( y = \sqrt{x} + 4 \)
   b) \( y = \sqrt{x} + 4 \)
   c) \( y = \sqrt{x - 4} - 3 \)

2. Starting with the graph of \( y = x^2 \), describe how you would sketch each of the following graphs.
   a) \( y = x^2 - 5 \)
   b) \( y = (x - 5)^2 \)
   c) \( y = (x + 5)^2 + 4 \)

3. Describe how you would find the domain and range of each of the following functions.
   a) \( y = x^2 + 4 \)
   b) \( y = x - 4 \)
   c) \( y = \sqrt{x - 5} - 3 \)

Practise

A

1. The function \( y = f(x) \) is given. Describe how the graphs of the following functions can be obtained from the graph of \( y = f(x) \).
   a) \( y = f(x) + 5 \)
   b) \( y = f(x) - 6 \)
   c) \( y = f(x - 4) \)
   d) \( y = f(x + 8) \)
   e) \( y = f(x) - 3 \)
   f) \( y = f(x) + 7 \)
   g) \( y = f(x + 3) - 5 \)
   h) \( y = f(x - 6) + 2 \)
   i) \( y = f(x - 5) - 7 \)
   j) \( y = f(x + 2) + 9 \)

2. The function \( y = f(x) \) has been transformed to \( y = f(x - h) + k \). Determine the values of \( h \) and \( k \) for each of the following transformations.
   a) 6 units upward
   b) 8 units downward
   c) 3 units to the right
   d) 5 units to the left
   e) 2 units to the left and 4 units downward
   f) 7 units to the right and 7 units upward

3. State the domain and range of each function.
   a) \( y = x + 2 \)
   b) \( y = x - 4 \)
   c) \( y = x^2 - 3 \)
   d) \( y = (x - 2)^2 \)
   e) \( y = (x + 5)^2 - 1 \)
   f) \( y = \sqrt{x + 1} \)
   g) \( y = \sqrt{x - 5} \)
   h) \( y = \sqrt{x - 3} + 6 \)
4. The graph of a function \( y = f(x) \) is shown. Sketch the graph of each of the following.
   a) \( y = f(x) - 4 \)  
   b) \( y = f(x) + 2 \)  
   c) \( y = f(x - 4) \)  
   d) \( y = f(x + 2) \)  
   e) \( y = f(x - 3) - 2 \)  
   f) \( y = f(x + 4) + 3 \)

5. The graph of the function drawn in blue is a translation image of the function drawn in red. Write an equation for each function drawn in blue. Check each equation using a graphing calculator.

6. Use transformations to sketch the graph of each of the following functions, starting with the graph of \( y = x \).
   a) \( y = x + 4 \)  
   b) \( y = x - 5 \)  
   c) \( y = (x - 4) \)  
   d) \( y = (x + 2) \)  
   e) \( y = (x + 5) - 2 \)  
   f) \( y = (x - 1) + 6 \)

7. Use transformations to sketch the graph of each of the following functions, starting with the graph of \( y = \sqrt{x} \).
   a) \( y = \sqrt{x + 7} \)  
   b) \( y + 3 = \sqrt{x} \)  
   c) \( y = \sqrt{x + 3} \)  
   d) \( y = \sqrt{x - 4} \)  
   e) \( y = \sqrt{x - 6} + 3 \)  
   f) \( y = \sqrt{x + 5} + 4 \)

8. Use transformations to sketch the graph of each of the following functions, starting with the graph of \( y = x^2 \).
   a) \( y = x^2 + 3 \)  
   b) \( y + 2 = x^2 \)  
   c) \( y = (x - 7)^2 \)  
   d) \( y = (x + 6)^2 \)  
   e) \( y = (x + 4)^2 - 3 \)  
   f) \( y = (x - 5)^2 + 5 \)
Apply, Solve, Communicate

9. Falling objects The approximate height above the ground of a falling object dropped from the top of a building is given by the function

\[ h(t) = -5t^2 + d \]

where \( h(t) \) metres is the height of the object \( t \) seconds after it is dropped, and \( d \) metres is the height from which it is dropped. The table shows the heights of three tall buildings in Canada.

<table>
<thead>
<tr>
<th>Building</th>
<th>Height (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Petro-Canada 1, Calgary</td>
<td>210</td>
</tr>
<tr>
<td>Two Bloor West, Toronto</td>
<td>148</td>
</tr>
<tr>
<td>Complexe G, Québec City</td>
<td>126</td>
</tr>
</tbody>
</table>

a) Write the three functions, \( f(P-C1) \), \( f(TBW) \), and \( f(CG) \), that describe the height of a falling object above the ground \( t \) seconds after it is dropped from the top of each building.

b) Graph \( h(t) \) versus \( t \) for the three functions on the same set of axes or in the same viewing window of a graphing calculator.

c) How could you transform the graph of \( f(P-C1) \) onto the graph of \( f(TBW) \)?

d) How could you transform the graph of \( f(CG) \) onto the graph of \( f(TBW) \)?

e) How could you transform the graph of \( f(P-C1) \) onto the graph of \( f(CG) \)?

10. Service calls Elena and Mario both repair kitchen appliances. Elena charges $45 for a service call, plus $35/h for labour. Mario charges $40 for a service call, plus $35/h for labour. Write an equation for the cost, \( C \) dollars, of a service call in terms of the number of hours worked, \( t \)

a) for Elena

b) for Mario

c) How are the graphs of the two equations related? Explain.

11. Communication When the graph of \( y = 2x + 3 \) is translated 1 unit to the right and 2 units upward, how is the resulting graph related to the graph of \( y = 2x + 3 \)? Explain.

12. Application Many companies pay their employees using a salary scale that depends on the number of years worked. One salary scale is modelled by the function \( S(y) = 25000 + 2250y \), where \( S(y) \) dollars is the salary and \( y \) is the number of years worked for the company. The employees’ union negotiates an increase of $1000 for each employee.

a) How is the graph of \( S(y) \) transformed by the increase?

b) Write the function that models the salary scale after the increase.

c) State a reasonable domain and range for the function in part b). Justify your reasoning.
13. **Greatest integer function** The greatest integer function is defined by \( [x] = \) the greatest integer that is less than or equal to \( x \). For example, \( [4] = 4 \), \( [4.83] = 4 \), and \( [-5.3] = -6 \). The graph of \( y = [x] \) is shown.

a) Explain the meanings of the open and closed dots on the graph of \( y = [x] \).

b) State the domain and range of \( y = [x] \).

c) Use transformations to sketch the graph of \( y = [x] + 2 \); \( y = [x - 3] \); \( y = [x + 4] - 1 \).

14. **Parking costs** EZ-Park determines its parking charges based on the greatest integer function \( y = [x + 2] + 3 \), where \( y \) is the parking charge, in dollars, and \( x \) is the number of hours that a vehicle is in the parking garage.

a) Sketch the graph of the function.

b) How much would a driver pay to park for 30 min? for 1 h? for 1 h 25 min? for 3 h 1 min?

15. What transformation relates the graph of

a) \( y = f(x - 4) \) to its image, \( y = f(x + 3) \)?

b) \( y = f(x) + 5 \) to its image, \( y = f(x) - 7 \)?

16. Describe a vertical translation that could be applied to the graph of \( y = \sqrt{x} \) so that the translation image passes through the point (4, 0).

17. **Inquiry/Problem Solving** The function \( y = x + 3 \) could be a vertical translation of \( y = x \) three units upward or a horizontal translation of \( y = x \) three units to the left. Explain why.

18. **Chemistry** a) One way to describe the concentration of an acid is as a percent by volume. For example, in 40 mL of a 30% acid solution, the volume of pure acid is \( 40 \times \frac{30}{100} \) or 12 mL, and the volume of water is 40 – 12 or 28 mL. If 5 mL of pure acid is mixed with 20 mL of water to give 25 mL of acid solution, the concentration of the solution is given by \( \frac{5}{25} \times 100\% = 20\% \). If water is mixed with 50 mL of 40% acid solution, write an equation that describes the acid concentration, \( C(x) \), as a function of the volume of water added, \( x \).

b) Graph \( C(x) \) versus \( x \).

c) What is the acid concentration after 10 mL of water have been added?
d) Write an equation that describes the acid concentration as a function of the volume of water added to 40 mL of 50% acid solution.
e) Graph \( C(x) \) versus \( x \) for the function from part d).
f) How could you transform the graph from part e) onto the graph from part b)?

CAREER CONNECTION  Veterinary Medicine

There are many more domestic animals in Canada than there are people. For example, in addition to the millions of dogs and cats in Canadian homes, there are over 12 000 000 cattle and 10 000 000 pigs on Canadian farms. Medical services for these and other animals are provided by workers in the field of veterinary medicine.

1. Ages of cats and dogs  As with humans, the medical needs of domestic animals change as they age. However, humans and domestic animals age differently. For a small dog, aged 3 years or more and with a mass up to about 11 kg, the number of human years equivalent to the age of the dog is given by the formula
\[
h(a) = 4a + 20
\]
where \( h \) is the equivalent number of human years, and \( a \) is the age of the dog. For a domestic cat aged 3 years or more, the number of human years equivalent to the age of the cat is given by the formula
\[
h(a) = 4a + 15
\]
where \( h \) is the equivalent number of human years, and \( a \) is the age of the cat.

a) Graph \( h \) versus \( a \) for cats and for small dogs over the domain 3 years to 15 years on the same axes or in the same viewing window of a graphing calculator.
b) Describe how the graphs are related by a transformation.
c) A cat and a small dog were born on the same day and are over 3 years old. How do the numbers of human years equivalent to their ages compare? Explain.
d) If their ages are expressed as equivalent human years, do cats and small dogs age at the same rate after the age of 3? Explain.
e) If their ages are expressed as equivalent human years, do cats and small dogs age at the same rate from birth? How do you know?

2. Research  Use your research skills to investigate

a) the training needed to become a veterinarian, also known as a doctor of veterinary medicine
b) the organizations that employ veterinarians
c) other careers that involve animal care