### 3.7 Combinations of Transformations

Because of its mathematical simplicity, the 3-4-5 right triangle has as much appeal today as it did thousands of years ago. In architecture, a 3-4-5 right triangle, with 4 as the base, has a hypotenuse with the slope of a comfortable stairway.

Some buildings have A-frame roofs. An example can be seen at the Alexander Graham Bell National Historic Site in Baddeck, Nova Scotia. For many A-frame roofs, a cross section is an isosceles triangle.

In some cases, the isosceles triangle can be formed by attaching two 3-4-5 right triangles. Two different isosceles triangles can be formed in this way.

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**INVESTIGATE & INQUIRE**

1. By graphing each function using a table of values, show the right triangle formed by each of the following functions, the $x$-axis, and the $y$-axis.

   a) \( y = -\frac{3}{4}x + 3, \ x \geq 0, \ y \geq 0 \)

   b) \( y = -\frac{4}{3}x + 4, \ x \geq 0, \ y \geq 0 \)

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<thead>
<tr>
<th>( x )</th>
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<tr>
<td>4</td>
<td></td>
<td>3</td>
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<td>2</td>
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2. What are the side lengths of the triangle in question 1a)?
3. What are the side lengths of the triangle in question 1b)?

4. What transformations must be applied to \( y = x \) to give the function \( y = -\frac{3}{4}x + 3 \)?

5. What transformations must be applied to \( y = x \) to give the function \( y = -\frac{4}{3}x + 4 \)?

6. In questions 4 and 5, must the transformations be applied in a particular order, or does the order have no effect on the result? Explain.

7. What transformation could be applied to the triangle from question 1a), so that the original triangle and its image form an isosceles triangle? Is there more than one answer? Explain.

8. Repeat question 7 for the triangle from question 1b).

9. State the dimensions and the height of each of the different isosceles triangles that can be formed in questions 7 and 8.

In this section, translations, expansions, compressions, and reflections will be used to perform combinations of transformations on functions. To simplify the procedure and give the desired results, perform the transformations in the following order.
• expansions and compressions
• reflections
• translations

In other words, perform multiplications (expansions, compressions, and reflections) before additions and subtractions (translations).

Example 1 Vertical Stretching and Reflecting

a) Copy the graph of \( y = f(x) \), as shown.
On the same set of axes, graph \( y = -3f(x) \) and \( y = -\frac{1}{2}f(x) \).

b) Describe how the graphs of \( y = -3f(x) \) and \( y = -\frac{1}{2}f(x) \) are related to the graph of \( y = f(x) \).
SOLUTION

a) Use the given graph to complete a table of values for the function \( y = f(x) \). Then, complete tables of values for the functions \( y = -3f(x) \) and \( y = -\frac{1}{2}f(x) \), and draw the graphs.

\[
\begin{array}{c|c|c|c|c|c}
 x & y & x & y & x & y \\
 4 & 0 & 4 & -3(0) = 0 & 4 & -\frac{1}{2}(0) = 0 \\
 2 & 3 & 2 & -3(3) = -9 & 2 & -\frac{1}{2}(3) = -1.5 \\
 0 & 0 & 0 & -3(0) = 0 & 0 & -\frac{1}{2}(0) = 0 \\
 -2 & 2 & -2 & -3(2) = -6 & -2 & -\frac{1}{2}(2) = -1 \\
 -4 & 0 & -4 & -3(0) = 0 & -4 & -\frac{1}{2}(0) = 0 \\
\end{array}
\]

b) The point \((x, y)\) on the graph of the function \( y = f(x) \) becomes the point \((x, -3y)\) on the graph of \( y = -3f(x) \).

The graph of \( y = -3f(x) \) is the graph of \( y = f(x) \) expanded vertically by a factor of 3 and reflected in the \(x\)-axis.

The point \((x, y)\) on the graph of the function \( y = f(x) \) becomes the point \((x, -\frac{1}{2}y)\) on the graph of \( y = -\frac{1}{2}f(x) \).

The graph of \( y = -\frac{1}{2}f(x) \) is the graph of \( y = f(x) \) compressed vertically by a factor of \( \frac{1}{2} \) and reflected in the \(x\)-axis.

Note that, in Example 1, the three functions have the same domain, \(-4 \leq x \leq 4\). The range of \( y = f(x) \) is \(0 \leq y \leq 3\), of \( y = -3f(x) \) is \(-9 \leq y \leq 0\), and of \( y = -\frac{1}{2}f(x) \) is \(-1.5 \leq y \leq 0\). The points \((4, 0)\), \((0, 0)\), and \((-4, 0)\) are invariant.
**Example 2 Transforming Quadratic Functions**

Sketch the graph of \( y = x^2 \) and the graph of \( y = \frac{1}{2} (x + 4)^2 - 5 \).

**Solution**

Sketch the graph of \( y = x^2 \).

To sketch the graph of \( y = \frac{1}{2} (x + 4)^2 - 5 \), first sketch the graph of \( y = \frac{1}{2} x^2 \). This graph is a vertical compression of \( y = x^2 \) by a factor of \( \frac{1}{2} \).

Then, apply the horizontal translation of 4 units to the left and the vertical translation of 5 units downward.

The result is the graph of \( y = \frac{1}{2} (x + 4)^2 - 5 \).

For such functions as \( y = (3x + 6)^2 \) and of \( y = \sqrt{-x + 5} \), factor the coefficient of the \( x \)-term to identify the characteristics of the function more easily.

The function \( y = (3x + 6)^2 \) becomes \( y = (3(x + 2))^2 \). Therefore, the graph of \( y = (3x + 6)^2 \) is the graph of \( y = x^2 \) compressed horizontally by a factor of \( \frac{1}{3} \) and translated 2 units to the left.

The function \( y = \sqrt{-x + 5} \) becomes \( y = \sqrt{x - 5} \). Therefore, the graph of \( y = \sqrt{-x + 5} \) is the graph of \( y = \sqrt{x} \) reflected in the \( y \)-axis and translated 5 units to the right.
**Example 3** Transforming Radical Functions

Given \( f(x) = \sqrt{x} \), sketch the graph of \( y = f(x) \) and the graph of \( y = 2f(-x - 3) + 4 \).

**Solution**

Sketch the graph of \( y = \sqrt{x} \).

The graph of \( y = 2 \cdot f(-x - 3) + 4 \) is the graph of \( y = 2 \cdot \sqrt{-x - 3} + 4 \).

Rewrite \( y = 2\sqrt{-x - 3} + 4 \) as \( y = 2\sqrt{-x + 3} + 4 \).

To sketch the graph of \( y = 2\sqrt{-x + 3} + 4 \), first sketch the graph of \( y = \sqrt{-x + 3} + 4 \). This graph is a vertical expansion of the graph of \( y = \sqrt{x} \) by a factor of 2.

Then, sketch the graph of \( y = 2\sqrt{-x} \), which is a reflection of the graph of \( y = \sqrt{-x} \) in the \( y \)-axis.

Then, apply the horizontal translation of 3 units to the left and the vertical translation of 4 units upward.

The result is the graph of \( y = 2f(-x - 3) \) or \( y = 2\sqrt{-x - 3} + 4 \).

**Example 4** Horizontal Stretching and Reflecting of Radical Functions

Compare the graphs of \( y = \sqrt{-x} \), \( y = \sqrt{-2x} \), and \( y = \sqrt{-\frac{1}{2}x} \) to the graph of \( y = \sqrt{x} \).

**Solution**

Complete tables of values using convenient values for \( x \), or use a graphing calculator.

\[
\begin{array}{c|c}
\hline
x & y \\
\hline
0 & 0 \\
1 & 1 \\
4 & 2 \\
9 & 3 \\
16 & 4 \\
\hline
\end{array}
\quad
\begin{array}{c|c}
\hline
x & y \\
\hline
0 & 0 \\
-1 & 1 \\
-4 & 2 \\
-9 & 3 \\
-16 & 4 \\
\hline
\end{array}
\quad
\begin{array}{c|c}
\hline
x & y \\
\hline
0 & 0 \\
-\frac{1}{2} & 1 \\
-2 & 2 \\
-\frac{9}{2} & 3 \\
-8 & 4 \\
\hline
\end{array}
\quad
\begin{array}{c|c}
\hline
x & y \\
\hline
0 & 0 \\
-2 & 1 \\
-8 & 2 \\
-18 & 3 \\
-32 & 4 \\
\hline
\end{array}
\]

For the function \( y = 2\sqrt{-x - 3} + 4 \), the domain is \( x \leq -3 \) and the range is \( y \geq 4 \).
The graph of \( y = \sqrt{-x} \) is the graph of \( y = \sqrt{x} \) reflected in the \( y \)-axis.

The graph of \( y = \sqrt{-2x} \) is the graph of \( y = \sqrt{x} \) compressed horizontally by a factor of \( \frac{1}{2} \) and reflected in the \( y \)-axis.

The graph of \( y = \sqrt{-\frac{1}{2}x} \) is the graph of \( y = \sqrt{x} \) expanded horizontally by a factor of 2 and reflected in the \( y \)-axis.

**Example 5 Writing Equations**

The graph of \( y = \sqrt{x} \) is expanded vertically by a factor of 5, reflected in the \( x \)-axis, and translated 6 units to the right and 3 units downward. Write the equation of the transformed function.

**Solution**

When \( y = \sqrt{x} \) is expanded vertically by a factor of 5, the function becomes \( y = 5\sqrt{x} \).

This function is reflected in the \( x \)-axis, becoming \( y = -5\sqrt{x} \).

The function \( y = -5\sqrt{x} \) is then translated 6 units to the right and 3 units downward to give \( y = -5\sqrt{x} - 6 - 3 \).

The equation of the transformed function is \( y = -5\sqrt{x} - 6 - 3 \).
**Example 6 Transforming Quadratic Functions**

Given \( f(x) = x^2 \), sketch the graph of \( y = f(x) \) and the graph of \( y = -f(2(x - 5)) + 6 \).

**Solution**

Sketch the graph of \( y = x^2 \).

The graph of \( y = -f(2(x - 5)) + 6 \) is the graph of \( y = -(2(x - 5))^2 + 6 \).

To sketch the graph of \( y = -(2(x - 5))^2 + 6 \), first sketch the graph of \( y = (2x)^2 \). This graph is a horizontal compression of the graph of \( y = x^2 \) by a factor of \( \frac{1}{2} \).

Then, sketch the graph of \( y = -(2x)^2 \), which is a reflection of the graph of \( y = (2x)^2 \) in the \( x \)-axis.

Then, apply the horizontal translation of 5 units to the right and the vertical translation of 6 units upward. The result is the graph of \( y = -f(2(x - 5)) + 6 \) or \( y = -(2(x - 5))^2 + 6 \).

**Key Concepts**

- Perform combinations of transformations in the following order.
  * expansions and compressions
  * reflections
  * translations
- If necessary, factor the coefficient of the \( x \)-term to identify the characteristics of a function more easily.

**Communicate Your Understanding**

1. Identify the combination of transformations on \( y = x^2 \) that results in the given function.
   a) \( y = -x^2 + 4 \)  
   b) \( y = -3(x - 1)^2 + 5 \)
2. Describe how you would graph the function \( y = 2\sqrt{x + 3} - 7 \).
3. Describe how you would graph the function \( y = \sqrt{-x - 4} \).
Practise

A

1. Describe how the graph of each of the following functions can be obtained from the graph of \( y = f(x) \).
   a) \( y = 2f(x) + 3 \)
   b) \( y = \frac{1}{2}f(x) - 2 \)
   c) \( y = f(x + 4) + 1 \)
   d) \( y = 3f(x - 5) \)
   e) \( y = f\left(\frac{x}{2}\right) - 6 \)
   f) \( y = f(2x) + 1 \)

2. Describe how the graph of each of the following functions can be obtained from the graph of \( y = f(x) \).
   a) \( y = -2f(x) \)
   b) \( y = -\frac{1}{3}f(x) \)
   c) \( y = f(-4x) \)
   d) \( y = f\left(-\frac{1}{2}x\right) \)

3. Describe how the graph of each of the following functions can be obtained from the graph of \( y = f(x) \).
   a) \( y = -f(2x) \)
   b) \( y = 3f(-2x) \)
   c) \( y = -\frac{1}{2}f\left(\frac{x}{3}\right) \)
   d) \( y = 4f(x - 6) + 2 \)
   e) \( y = -2f(x) - 3 \)
   f) \( y = -f(x - 3) + 1 \)
   g) \( y = 3f(2x) - 6 \)
   h) \( y = \frac{1}{2}f\left(\frac{1}{2}x\right) - 4 \)

4. The graph of \( y = f(x) \) is shown.

5. The graph of \( y = f(x) \) is shown. Sketch the graph of each of the following functions, state its domain and range, and identify any invariant points.
   a) \( y = f(x - 4) + 2 \)
   b) \( y = f(x + 2) - 4 \)
   c) \( y = \frac{1}{3}f(x) - 3 \)
   d) \( y = f(2x) + 3 \)
   e) \( y = -2f(x) \)
   f) \( y = f(-x) - 2 \)
   g) \( y = f\left(-\frac{1}{2}x\right) \)
   h) \( y = -\frac{1}{2}f(-2x) \)

6. Sketch each set of functions on the same set of axes in the given order.
   a) \( y = x \)
   b) \( y = x^2 \)
   c) \( y = 2x + 2 \)
   d) \( y = \frac{1}{2}x^2 \)
   e) \( y = \sqrt{x} \)
   f) \( y = \sqrt{x} \)
   \( y = 2\sqrt{x} \)
   \( y = -\sqrt{2x} \)
   \( y = 2\sqrt{x} \)
   \( y = -\sqrt{2x} \)
   \( y = 2\sqrt{x} \)
   \( y = 2\sqrt{x} - 3 \)
   \( y = 2\sqrt{x} + 5 \)
7. Describe how the graph of each of the following functions can be obtained from the graph of \( y = f(x) \).
   \[ \text{a)} \quad y = f(2(x - 4)) \quad \text{b)} \quad y = f(-(x + 1)) - 1 \]
   \[ \text{c)} \quad y = f(3(x + 4)) + 5 \quad \text{d)} \quad y = -2f(4(x - 2)) \]
   \[ \text{e)} \quad y = f(-x + 2) \quad \text{f)} \quad y = f(2x + 8) - 4 \]
   \[ \text{g)} \quad y = f(4 - x) + 5 \quad \text{h)} \quad y = f(3x - 6) + 8 \]

8. Given \( f(x) = x^2 \), sketch the graph of each of the following, and state the domain and range.
   \[ \text{a)} \quad y = f(x - 3) + 1 \quad \text{b)} \quad y = 2f(x + 5) - 4 \]
   \[ \text{c)} \quad y = \frac{1}{2} f\left(\frac{1}{2} x\right) + 3 \quad \text{d)} \quad y = -2f(2x - 2) - 3 \]
   \[ \text{e)} \quad y = f(3 - x) + 2 \quad \text{f)} \quad y = -\frac{1}{2} f(2x + 6) - 2 \]

9. Given \( f(x) = \sqrt{x} \), sketch the graph of each of the following, and state the domain and range.
   \[ \text{a)} \quad y = f(x - 5) - 4 \quad \text{b)} \quad y = 3f(x + 3) + 2 \]
   \[ \text{c)} \quad y = \frac{1}{2} f(2(x - 1)) - 2 \quad \text{d)} \quad y = 2f(3x - 9) + 1 \]
   \[ \text{e)} \quad y = -f(-x) + 5 \quad \text{f)} \quad y = -2f(4 - x) - 3 \]

10. **Technology** The calculator display shows the graph of \( y = (x + 2)^2 + 3 \) and its image after a reflection in the \( x \)-axis and a reflection in the \( y \)-axis. Write the equation of the image.

11. **Technology** The graph of \( y = x^2 \) is expanded vertically by a factor of 3, translated 4 units to the right, and translated 2 units downward. Write the equation of the transformed function. Check your solution using a graphing calculator.

12. **Technology** The graph \( y = \sqrt{x} \) is expanded horizontally by a factor of 2, reflected in the \( y \)-axis, and translated 6 units to the right. Write the equation of the transformed function. Check your solution using a graphing calculator.

**Apply, Solve, Communicate**

13. **Ski chalet** A cross section of the roof of a ski chalet can be modelled by the following two functions and the \( x \)-axis.
   \[ y = \frac{5}{3} x + 10, \quad -6 \leq x \leq 0, \quad 0 \leq y \leq 10 \]
   \[ y = -\frac{5}{3} x + 10, \quad 0 \leq x \leq 6, \quad 0 \leq y \leq 10 \]
   \[ \text{a)} \quad \text{Graph both functions on the same axes or in the same viewing window.} \]
   \[ \text{b)} \quad \text{Find the side lengths and the height of the isosceles triangle formed by the two functions and the \( x \)-axis. Round to the nearest tenth of a unit, if necessary.} \]
   \[ \text{c)} \quad \text{What transformations must be applied to } y = x \text{ to give the function } y = \frac{5}{3} x + 10? \text{ to give the function } y = -\frac{5}{3} x + 10? \]
14. **Application** The height, $y$ metres, of an emergency flare fired upward from a small boat can be modelled by the function

$$y = -5(x - 4)^2 + 80$$

where $x$ seconds is the time since the flare was fired.

a) Describe how the graph of $y = -5(x - 4)^2 + 80$ can be obtained by transforming the graph of $y = x^2$.

b) Interpret the equation of the transformed function to find the maximum height reached by the flare and the time it takes to reach this height.

15. Use transformations and the zeros of the quadratic function

$$f(x) = (x + 2)(x - 6)$$

to determine the zeros of each of the following functions.

a) $y = -2f(x)$

b) $y = f(-2x)$

c) $y = 2f(-x)$

d) $y = -\frac{1}{2}f\left(\frac{1}{2}x\right)$

e) $y = -f(x + 1)$

f) $y = f(-x - 2)$

16. **Communication** The graph of $f(x) = \sqrt{16 - x^2}$ is shown.

Sketch the graph of each of the following functions, and state its domain and range.

a) $y = -\frac{1}{2}f(x)$

b) $y = f(-2x)$

c) $y = \frac{1}{2}f(x) + 5$

d) $y = -\frac{1}{2}f(x) - 3$

e) $y = (f(x) - 3) - 2$

f) $y = -2f(x + 6) + 5$

17. The function $y = f(x)$ has been transformed to $y = af(k(x - h)) + q$.

Determine the values of $a$, $k$, $h$, and $q$ for each of the following transformations.

a) a vertical expansion by a factor of 3 and a reflection in the $y$-axis

b) a vertical compression by a factor of $\frac{1}{3}$, a horizontal expansion by a factor of 2, a translation of 6 units to the right, and a translation of 1 unit downward

c) a vertical expansion by a factor of 2, a horizontal compression by a factor of $\frac{1}{2}$, a reflection in the $x$-axis, a reflection in the $y$-axis, a translation of 7 units to the left, and a translation of 4 units upward

18. **Falling objects** On the Earth, if an object is dropped from an initial height of $x$ metres, its approximate height above the ground, $h$ metres, after $t$ seconds, is given by $h(t) = -5t^2 + x$. On the moon, the approximate height is given by $h(t) = -0.8t^2 + x$. For objects dropped from an initial height of 20 m, the functions are $h(t) = -5t^2 + 20$ and $h(t) = -0.8t^2 + 20$.  

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a) Translate the graph of \( h(t) = -5t^2 + 20 \) upward by 105 units. Explain the meaning of the point that the resulting graph and the graph of \( h(t) = -0.8t^2 + 20 \) have in common.

b) The approximate height of an object dropped on Jupiter is given by \( h(t) = -12.8t^2 + x \). So, for an initial height of 320 m, the function is \( h(t) = -12.8t^2 + 320 \). Graph \( h(t) \) versus \( t \) for \( h(t) = -12.8t^2 + 320 \) and \( h(t) = -0.8t^2 + 20 \) on the same axes or in the same viewing window.

c) **Inquiry/Problem Solving** Describe how the graph of \( h(t) = -0.8t^2 + 20 \) can be transformed onto the graph of \( h(t) = -12.8t^2 + 320 \). Justify your reasoning.

d) Explain the meaning of the point that the two graphs in part c) have in common.

e) State the domain and range of each function in part c).

**C**

19. a) Describe how the graph of \( y = 2\sqrt{x} + 1 \) can be obtained from the graph of \( y = \sqrt{x} \).

b) Describe how the graph of \( y = \sqrt{4x + 4} \) can be obtained from the graph of \( y = \sqrt{x} \).

c) How are the graphs of \( y = 2\sqrt{x} + 1 \) and \( y = \sqrt{4x + 4} \) related? Explain.

20. Describe how the graph of the function \( y = -\sqrt{x} + 2 - 3 \) can be obtained from the graph of \( y = x^2, x \leq 0 \).

21. a) Expand the graph of \( y = x \) vertically by a factor of 2. Then, translate the result 3 units to the left and 5 units downward. Compare the result to the graph of \( y = 2x + 1 \). Explain your findings.

b) If the same transformations are performed on the graph of \( y = x^2 \), instead of \( y = x \), is the result the same as the graph of \( y = 2x^2 + 1 \)? Explain.

### Achievement Check

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<th>Thinking/Inquiry/Problem Solving</th>
<th>Communication</th>
<th>Application</th>
</tr>
</thead>
</table>

- **Given:**
  - A horizontal line segment from \((0, 0)\) to \((2, 0)\)
  - A vertical line segment from \((0, 0)\) to \((0, 2)\)
  - A semi-circle centred at \((1, 0)\) with endpoints \((0, 0)\) and \((0, 2)\) and passing through point \((1, 1)\), radius 1 unit

a) Sketch the lower case letters of the alphabet that are possible to print using transformations of the given figures.

b) What other types of transformations would be needed to print the other letters?