4.3 The Sine Law and the Cosine Law

The Peace Tower is the tallest part of Canada's Parliament Buildings. A bronze mast, which flies the Canadian flag, stands on top of the Peace Tower.

From a point 25 m from the foot of the tower, the angle of elevation of the top of the tower is 74.8°. From the same point, the angle of elevation of the top of the mast is 76.3°.

**Investigate & Inquire**

To find the height of the mast, use the diagram shown. The given information is marked on the diagram. DC is the height of the mast, and BC is the height of the tower.

1. Find the height of the mast, to the nearest tenth of a metre, using only right triangles.

2. **a)** List measurements you would use to find the height of the mast using the cosine law in \(\triangle ACD\).
   
   **b)** Find these measurements.
   
   **c)** Use the cosine law to find the height of the mast, to the nearest tenth of a metre.

3. **a)** List measurements you would use to find the height of the mast using the sine law in \(\triangle ACD\).
   
   **b)** Find these measurements.
   
   **c)** Use the sine law to find the height of the mast, to the nearest tenth of a metre.

4. Compare your answers from questions 1, 2, and 3. Which method did you prefer? Explain.
You have previously applied the sine law and the cosine law to acute triangles. You have seen that the sine law and the cosine law also apply to obtuse triangles.

The sine law for acute and obtuse triangles can be developed as follows.

In $\triangle ABC$, draw $AD$ perpendicular to $BC$, or to $BC$ extended. $AD$ is the altitude or height, $h$, of $\triangle ABC$.

Acute Triangle

$$h = b \sin C$$

Obtuse Triangle

$$h = b \sin C$$

Recall that $\sin (180^\circ - \theta) = \sin \theta$.

In $\triangle ACD$, $\frac{h}{b} = \sin C$ and $\frac{h}{c} = \sin B$.

For both the acute and the obtuse triangles, $\frac{bc}{c} = \frac{b}{c}$ and $\frac{bc}{b} = \frac{c}{b}$.

Divide both sides by $bc$:

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

By drawing the altitude from $C$, we can similarly show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Combining the results gives the following forms of the sine law.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$
**Example 1 The Sine Law, Given Two Angles and a Side**

In \(\triangle RST\), \(\angle S = 40^\circ\), \(\angle T = 21^\circ\), and \(r = 46\) cm. Find \(t\), to the nearest centimetre.

**Solution**

Draw a diagram.

Find the measure of \(\angle R\).

\[
\angle R = 180^\circ - 40^\circ - 21^\circ = 119^\circ
\]

Use the sine law to find \(t\).

\[
\frac{t}{\sin T} = \frac{r}{\sin R}
\]

\[
t = \frac{46 \sin 21^\circ}{\sin 119^\circ}
\]

So, \(t \approx 19\) cm, to the nearest centimetre.

**Example 2 The Sine Law, Given Two Sides and the Angle Opposite One of Them**

In \(\triangle PQR\), \(\angle P = 105.2^\circ\), \(p = 23.2\) cm, and \(r = 18.5\) cm. Solve the triangle, rounding the side length to the nearest tenth of a centimetre and the angles to the nearest tenth of a degree, if necessary.

**Solution**

Draw a diagram.

Use the sine law to find the measure of \(\angle R\).

\[
\frac{\sin R}{r} = \frac{\sin P}{p}
\]

\[
\frac{\sin R}{18.5} = \frac{\sin 105.2^\circ}{23.2}
\]

\[
\sin R = \frac{18.5 \sin 105.2^\circ}{23.2}
\]

\(\angle R \approx 50.3^\circ\)
Find the measure of $\angle Q$.

$\angle Q = 180^\circ - 105.2^\circ - 50.3^\circ = 24.5^\circ$

Use the sine law to find $q$.

$$\frac{q}{\sin 24.5^\circ} = \frac{23.2}{\sin 105.2^\circ}$$

$$q = \frac{23.2\sin 24.5^\circ}{\sin 105.2^\circ} \approx 10.0$$

In $\triangle PQR$, $\angle R = 50.3^\circ$, $\angle Q = 24.5^\circ$, and $q = 10.0$ cm.

The cosine law for acute and obtuse triangles can be developed as follows.

In $\triangle ABC$, draw $AD$ perpendicular to $BC$, or to $BC$ extended.

$AD$ is the altitude or height, $h$, of $\triangle ABC$.

Acute Triangle

$$\text{In } \triangle AD \ C, \quad \frac{x}{b} = \cos C$$

$$x = b \cos C$$

and $b^2 = h^2 + x^2$

$$\text{In } \triangle ABD, \ c^2 = h^2 + (a - x)^2$$

$$= h^2 + a^2 - 2ax + x^2$$

$$= a^2 + (h^2 + x^2) - 2ax$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$
In \( \triangle AD \), \( \frac{x}{b} = \cos (180° - C) \)

\[
x = b \cos (180° - C) = -b \cos C
\]

and \( b^2 = h^2 + x^2 \)

In \( \triangle ABD \), \( c^2 = h^2 + (a + x)^2 \\
= h^2 + a^2 + 2ax + x^2 \\
= a^2 + b^2 + 2a(-b \cos C) \\
= a^2 + b^2 - 2ab \cos C
\]

The forms of the cosine law are as follows.

\[
a^2 = b^2 + c^2 - 2bc \cos A \\
b^2 = a^2 + c^2 - 2ac \cos B \\
c^2 = a^2 + b^2 - 2ab \cos C
\]

**EXAMPLE 3 The Cosine Law, Given Three Sides**

In \( \triangle ABC \), \( a = 9.6 \text{ m} \), \( b = 20.6 \text{ m} \), and \( c = 14.7 \text{ m} \). Solve the triangle. Round each angle measure to the nearest tenth of a degree.
**Solution**

Draw a diagram.

Use the cosine law to find the measure of an angle.

\[
\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{9.6^2 + 14.7^2 - 20.6^2}{2(9.6)(14.7)}
\]

\[
\angle B = 114.3^\circ
\]

Use the sine law to find the measure of \( \angle A \).

\[
\frac{\sin A}{a} = \frac{\sin B}{b}
\]

\[
\sin A = \frac{9.6 \sin 114.3^\circ}{20.6}
\]

\[
\angle A = 25.1^\circ
\]

Find \( \angle C \).

\[
\angle C = 180^\circ - 114.3^\circ - 25.1^\circ = 40.6^\circ
\]

In \( \triangle ABC \), \( \angle A = 25.1^\circ \), \( \angle B = 114.3^\circ \), and \( \angle C = 40.6^\circ \).

**Example 4** The Cosine Law, Given Two Sides and the Contained Angle

Find the length of \( CD \), to the nearest tenth of a metre.
**Solution**

Use $\triangle DEF$ and the cosine law to find the length of $DE$.

\[
DE^2 = DF^2 + EF^2 - 2(DF)(EF)\cos F
\]

\[
= 3.9^2 + 2.5^2 - 2(3.9)(2.5)\cos 97.4°
\]

$DE \approx 4.9$

Use $\triangle CED$ to find the measure of $\angle C$.

\[
\angle C = 180° - 56.7° - 48.2°
\]

$\angle C = 75.1°$

Use $\triangle CED$ and the sine law to find the length of $CD$.

\[
\frac{CD}{\sin 56.7°} = \frac{4.9}{\sin 75.1°}
\]

\[
CD = \frac{4.9\sin 56.7°}{\sin 75.1°}
\]

$CD \approx 4.2$

$CD = 4.2$ m, to the nearest tenth of a metre.

**Key Concepts**

- The forms of the sine law are
  \[
  \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
  \]
- The sine law can be used to solve any triangle when given
  - a) the measures of two angles and any side
  - b) the measures of two sides and the angle opposite one of those sides
- The forms of the cosine law are
  \[
  a^2 = b^2 + c^2 - 2bc\cos A \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}
  \]
  \[
  b^2 = a^2 + c^2 - 2accos B \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}
  \]
  \[
  c^2 = a^2 + b^2 - 2ab\cos C \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}
  \]
- The cosine law can be used to solve any triangle when given
  - a) the measures of two sides and the contained angle
  - b) the measures of three sides
Communicate Your Understanding

1. Describe how you would solve each of the following triangles. Justify your chosen method.

   a) \[ \triangle ABC \]
      - \( A \): \( 18 \) cm
      - \( B \): \( 18 \) m
      - \( C \): \( 37^\circ \)

   b) \[ \triangle DEF \]
      - \( D \): \( 7 \) m
      - \( E \): \( 8 \) m
      - \( F \): \( 14 \) m

   c) \[ \triangle TUS \]
      - \( T \): \( 18 \) m
      - \( U \): \( 33 \) m
      - \( S \): \( 114^\circ \)

   d) \[ \triangle XYZ \]
      - \( X \): \( 10 \) cm
      - \( Y \): \( 123^\circ \)
      - \( Z \): \( 12 \) cm

2. Explain why you cannot start with the sine law to solve \( \triangle XYZ \).

3. Explain why you cannot start with the cosine law to solve \( \triangle KLM \).

Practise

A

1. Find the length of the indicated side, to the nearest tenth.

   a) \[ \triangle RST \]
      - \( R \): \( 15.2 \) cm
      - \( S \): \( 51^\circ \)
      - \( T \): \( 63^\circ \)

   b) \[ \triangle EFG \]
      - \( E \): \( 31.2 \) cm
      - \( F \): \( 27.5 \) cm
      - \( G \): \( 82^\circ \)

   c) \[ \triangle ABC \]
      - \( A \): \( 112^\circ \)
      - \( B \): \( 11 \) cm
      - \( C \): \( 4.8 \) cm

   d) \[ \triangle EFD \]
      - \( E \): \( 8.3 \) cm
      - \( F \): \( 75.9^\circ \)
      - \( D \): \( 60.2^\circ \)
2. Find the measure of the indicated angle, to the nearest tenth of a degree.

a) \( \angle J = 49.7° \)

b) \( \angle L = 21.2° \)

c) \( \angle X = 19.8° \)

d) \( \angle M = 101° \)

e) \( \angle K = 24.2° \)

f) \( \angle P = 31.6° \)

3. Solve each triangle. Round answers to the nearest tenth, if necessary.

a) \( \triangle ABC, \angle A = 84°, \angle C = 40°, a = 5.6 \text{ m}. \)

b) \( \triangle PQR, \angle R = 28.5°, p = 10.4 \text{ cm}, r = 6.3 \text{ cm}. \)

c) \( \triangle LMN, \angle M = 62°, l = 16.9 \text{ cm}, n = 15.1 \text{ m}. \)

d) \( \triangle VWU, \angle W = 123.9°, \angle V = 22.2°, v = 27.5 \text{ km}. \)

e) \( \triangle XYZ, \angle X = 92.3°, y = 3.1 \text{ cm}, z = 2.8 \text{ cm}. \)

f) \( \triangle FGH, f = 12.6 \text{ m}, g = 8.5 \text{ m}, h = 6.3 \text{ m}. \)

4. Find the length of the indicated side, to the nearest tenth.

a) \( \triangle ABD, \angle A = 73°, a = 31.6 \text{ cm}, b = 13 \text{ m}. \)

b) \( \triangle LKM, \angle K = 27.3°, \angle M = 136°, l = 17.5 \text{ cm}, m = 19.8 \text{ cm}. \)

c) \( \triangle YXZ, \angle Y = 49.7°, \angle Z = 18.1°, y = 17.5 \text{ cm}, z = 19.8 \text{ cm}. \)

d) \( \triangle PQR, \angle P = 35°, \angle R = 125°, p = 2.7 \text{ mm}, r = 3.8 \text{ mm}. \)
Apply, Solve, Communicate

6. Solve $\triangle ABC$. Round answers to the nearest tenth.

7. **Measurement** An isosceles triangle has two 5.5-cm sides and two $32.4^\circ$ angles. Find
   a) the perimeter of the triangle, to the nearest tenth of a centimetre
   b) the area of the triangle, to the nearest tenth of a square centimetre

8. **Inquiry/ Problem Solving** Airport X is 150 km east of airport Y. An aircraft is 240 km from airport Y, and $23^\circ$ north of due west from airport Y. How far is the aircraft from airport X, to the nearest kilometre?
9. **Application** To determine the height of the Peace Tower on Parliament Hill in Ottawa, measurements were taken from a baseline AB. It was found that AB = 50 m, \( \angle XAY = 42.6^\circ \), \( \angle XAB = 60^\circ \), and \( \angle ABX = 81.65^\circ \). Calculate the height of the Peace Tower, to the nearest metre.

10. **Ship navigation** Two ships left Port Hope on Lake Ontario at the same time. One travelled at 12 km/h on a course of 235°. The other travelled at 15 km/h on a course of 105°. How far apart were the ships after four hours, to the nearest kilometre?

11. **Measurement** Find the area of \( \triangle XYZ \), to the nearest tenth of a square metre.

12. **Communication** a) Use the cosine law to find \( x \), to the nearest tenth.
   b) Use the Pythagorean theorem to find \( x \), to the nearest tenth.
   c) Explain why the two methods give the same results in a right triangle.

13. **Sine law in right triangles** Right \( \triangle ABC \) is shown. Write each of the ratios \( \frac{a}{\sin A}, \frac{b}{\sin B}, \) and \( \frac{c}{\sin C} \) in terms of \( a, b, \) or \( c, \) and verify that \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \) for a right triangle.
14. **Stikine Canyon** The Stikine Canyon in central British Columbia is often referred to as Canada's Grand Canyon. Two points X and Y are sighted from a baseline AB of length 30 m on the opposite side of the canyon. The angle measurements recorded from positions A and B were $\angle XAY = 31.3^\circ$, $\angle XBY = 18.5^\circ$, $\angle ABX = 25.6^\circ$, and $\angle BAY = 27.9^\circ$. Find the distance from X to Y, to the nearest metre.

15. **Geometry** Use the cosine law to show that opposite angles in a parallelogram are congruent.

16. **Measurement** In $\triangle RST$, $RS = 4.9$ m, $ST = 3.7$ m, and $RT = 8.1$ m. Find the area of $\triangle RST$, to the nearest tenth of a square metre.

17. **Measurement** In $\triangle ABC$, $BC = 46$ m, $\angle A = 42.2^\circ$, and $\angle B = 39.5^\circ$. Find the area of $\triangle ABC$, to the nearest tenth of a square metre.

18. **Measurement** Find the volume of the right prism, to the nearest cubic centimetre.

19. **Measurement** Find the volume of the right prism, to the nearest cubic metre.

20. **Analytic geometry** $\triangle PQR$ has vertices $P(1, 5)$, $Q(6, -7)$, and $R(-2, 1)$. Find the angle measures, to the nearest tenth of a degree.

**Achievement Check**

An equilateral triangle $ABC$ has been creased and folded so that its vertex A now rests on BC at D, such that $BD = 1$ and $DC = 2$. Find the length of

a) $AP$  

b) $AQ$  

c) $PQ$
CAREER CONNECTION Surveying

Surveying is the scientific measurement of natural or artificial features on the Earth’s surface. Surveyors are involved in a wide variety of tasks that require very accurate knowledge of locations. The distances and angles determined by surveyors are used in many ways, including drawing maps, positioning buildings and other structures correctly, and defining the property lines that separate one piece of land from another.

Because Canada is the world’s second-largest country, surveying Canada has been an enormous task. For example, it took almost 60 years to complete a survey of the Canada-U.S border, part of which runs through four of the Great Lakes. As a result of over 150 years of surveying work, detailed maps now exist for all parts of Canada.

1. From point P, the distance to one end of a pond is 450 m and the distance to the other end is 520 m. The angle formed by the lines of sight is 115°. Find the length of the pond, to the nearest ten metres.

2. Research Use your research skills to investigate the following.
   a) the education and training required to become a surveyor, and the organizations that employ surveyors
   b) the use of different types of surveying equipment, including manually controlled, electronic, and photographic instruments, and the use of satellite technology
   c) the work of the Geological Survey of Canada in exploring and mapping the country

PATTERN Power

a) Copy and complete the pattern.
   \[11^2 = \square\]
   \[101^2 = \square\]
   \[1001^2 = \square\]

b) Describe the pattern in words.

c) Explain why the pattern works.

d) Write the next 2 lines of the pattern.

e) Use the pattern to find \(\sqrt{100\,000\,020\,000\,001}\).