6.3 Geometric Sequences

In the sequence 2, 10, 50, 250, … , each term after the first is found by multiplying the preceding term by 5. Therefore, the ratio of consecutive terms is a constant.

\[
\frac{10}{2} = 5, \quad \frac{50}{10} = 5, \quad \frac{250}{50} = 5
\]

This type of sequence is called a geometric sequence. The ratio of consecutive terms is called the common ratio. For the sequence 2, 10, 50, 250, … , the common ratio is 5.

In a geometric sequence, the first term, \( t_1 \), is denoted by the letter \( a \).

An acoustic piano has 88 keys. Each key plays a note with a different frequency. The frequency of a note is measured in hertz, symbol Hz. One hertz is one vibration per second. The first and lowest note is assigned the letter A. It has a frequency of 27.5 Hz.

There are eight As on a piano. The second A has a frequency two times the frequency of the first A, or 55 Hz. The third A has a frequency two times the frequency of the second A, or 110 Hz. Each subsequent A has a frequency two times the frequency of the A that precedes it.

The first four As have the frequencies 27.5 Hz, 55 Hz, 110 Hz, and 220 Hz. The numbers 27.5, 55, 110, and 220 form a geometric sequence.

1. Copy and complete the table for this sequence.

<table>
<thead>
<tr>
<th>Note</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( A_4 )</th>
<th>( A_5 )</th>
<th>( A_6 )</th>
<th>( A_7 )</th>
<th>( A_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (Hz)</td>
<td>27.5</td>
<td>55</td>
<td>110</td>
<td>220</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Frequency (Hz) Expressed Using 27.5 and Powers of 2</td>
<td>27.5</td>
<td>( 27.5 \times 2^1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Frequency Expressed Using ( a ) and ( r )</td>
<td>( a )</td>
<td>( a \times r )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. What are the values of a and r for this sequence?

3. When you write an expression for a term using the letters a and r, you are writing a formula for the term. What is the formula for \( t_6 \)? \( t_7 \)? \( t_8 \)?

4. Evaluate \( t_6 \), \( t_7 \), and \( t_8 \).

5. Write the formula for the nth term of this geometric sequence.

6. There are seven G's on a piano. The frequency doubles from one G to the next. The lowest G has a frequency of about 49 Hz. Find the frequency of the highest G.

7. Describe the similarities and differences in the formulas for the nth term of an arithmetic sequence and the nth term of a geometric sequence.

8. Write the formula for the nth term of the following sequence.
2, –10, 50, –250, ...

**Example 1** Writing Terms of a Sequence

Given the formula for the nth term of a geometric sequence, \( t_n = 5(-2)^{n-1} \), write the first 5 terms.

**Solution 1** Paper-and-Pencil Method

\[
\begin{align*}
t_1 &= 5(-2)^{1-1} = 5(1) = 5 \\
t_2 &= 5(-2)^{2-1} = 5(-2) = -10 \\
t_3 &= 5(-2)^{3-1} = 5(4) = 20 \\
t_4 &= 5(-2)^{4-1} = 5(-8) = -40 \\
t_5 &= 5(-2)^{5-1} = 5(16) = 80
\end{align*}
\]

The first 5 terms are 5, –10, 20, –40, and 80.

**Solution 2** Graphing-Calculator Method

Adjust the mode settings to the Seq (sequence) graphing mode. Use the sequence function from the LIST OPS menu to generate the first 5 terms.

The first 5 terms are 5, –10, 20, –40, and 80.
**Example 2 Determining the Value of a Term**

Given the formula for the $n$th term, find $t_6$.

a) $t_n = 3(2)^{n-1}$

b) $f(n) = -5(4)^{n-1}$

**Solution 1  Paper-and-Pencil Method**

a) $t_n = 3(2)^{n-1}$

\[
t_6 = 3(2)^{6-1} = 3(2)^5 = 3(32) = 96
\]

b) $f(n) = -5(4)^{n-1}$

\[
f(6) = -5(4)^{6-1} = -5(4)^5 = -5(1024) = -5120
\]

**Solution 2  Graphing-Calculator Method**

Adjust the mode settings to the Seq (sequence) graphing mode. Use the sequence function from the LIST OPS menu to generate the 6th term.

a) $t_n = 3(2)^{n-1}$

b) $f(n) = -5(4)^{n-1}$

The general geometric sequence is $a, ar, ar^2, ar^3, \ldots$, where $a$ is the first term and $r$ is the common ratio.

\[
t_1 = a \\
t_2 = ar \\
t_3 = ar^2 \\
\vdots \\
t_n = ar^{n-1}, \text{ where } n \text{ is a natural number, and } r \neq 0.
\]

Note that $r$ is the ratio of any successive pair of terms. For example,

\[
\frac{t_2}{t_1} = \frac{ar}{a} = r \\
\frac{t_3}{t_2} = \frac{ar^2}{ar} = r
\]
Example 3 Finding the Formula for the $n$th Term

Find the formula for the $n$th term, $t_n$, and find $t_6$ for the geometric sequence $2, 6, 18, ...$

Solution

For the given sequence, $a = 2$ and $r = 3$.

$t_n = ar^{n-1}$

Substitute known values: $a = 2, r = 3$

The formula for the $n$th term is $t_n = 2(3)^{n-1}$.

Three ways to find $t_6$ are as follows.

Method 1: $t_6 = 2(3)^6 - 1$

Method 2: $t_6 = 2(3)^5$

Method 3: Use a graphing calculator.

So, $t_n = 2(3)^{n-1}$ and $t_6 = 486$.

Example 4 Finding the Number of Terms

Find the number of terms in the geometric sequence $3, 6, 12, ... , 384$.

Solution

For the given sequence, $a = 3$, $r = 2$, and $t_n = 384$.

Substitute the known values in the formula for the general term and solve for $n$.

Substitute known values: $a = 3, r = 2, t_n = 384$

Divide both sides by 3:

Simplify:

Write $128$ as a power of 2:

Equate the exponents:

Solve for $n$:

There are 8 terms in the sequence.
EXAMPLE 5  Finding $t_n$ Given Two Terms

In a geometric sequence, $t_5 = 1875$ and $t_7 = 46,875$. Find the first three terms of the sequence and $t_n$.

**Solution**

Substitute known values in the formula for the $n$th term to write a system of equations. Then, solve the system.

$$t_n = ar^{n-1}$$

Write an equation for $t_5$: $1875 = ar^4$ \hspace{1cm} (1)

Write an equation for $t_7$: $46,875 = ar^6$ \hspace{1cm} (2)

Divide (2) by (1):

$$\frac{46,875}{1875} = r^2$$

Solve for $r$: \pm 5 = r

Since $r = 5$ or $r = -5$, there are two possible solutions.

Substitute 5 for $r$ in (1): $1875 = a(5)^4$ \hspace{1cm} Substitute -5 for $r$ in (1): $1875 = a(-5)^4$

Solve for $a$: $1875 = 625a$ \hspace{1cm} 1875 = 625a

Since $a = 3$ and $r = 5$, the first three terms are 3, 15, and 75.

Since $a = 3$ and $r = -5$, the first three terms are 3, -15, and 75.

Substitute 3 for $a$ and 5 for $r$ in the formula for the $n$th term.

$t_n = 3(5)^{n-1}$

Substitute 3 for $a$ and -5 for $r$ in the formula for the $n$th term.

$t_n = 3(-5)^{n-1}$

So, the first three terms are 3, 15, and 75, and $t_n = 3(5)^{n-1}$, or the first three terms are 3, -15, and 75, and $t_n = 3(-5)^{n-1}$.
Key Concepts

- The general geometric sequence is \( a, ar, ar^2, ar^3, \ldots \), where \( a \) is the first term and \( r \) is the common ratio.
- The formula for the \( n \)th term, \( t_n \) or \( f(n) \), of a geometric sequence is \( t_n = ar^{n-1} \), where \( n \) is a natural number.

Communicate Your Understanding

1. Given a sequence of numbers written in order from least to greatest, explain how you would determine if the sequence is arithmetic, geometric, or neither.
2. Given the formula for the \( n \)th term of a geometric sequence, \( t_n = 4(3)^{n-1} \), describe how you would find the first 5 terms.
3. a) Describe how you would find the formula for the \( n \)th term of the geometric sequence 4, 8, 16, 32, ...
b) Describe how you would find \( t_{12} \) for this sequence.
4. Describe how you would find the number of terms in the sequence 5, 10, 20, ..., 1280.
5. Given that \( t_3 = 28 \) and \( t_4 = 56 \) for a geometric sequence, describe how you would find \( t_n \) for the sequence.

Practise

A

1. Determine whether each sequence is arithmetic, geometric, or neither. Then, find the next two terms of each sequence.
   a) 1, 4, 9, 16, ...
   b) 1, 2, 4, 8, ...
   c) 7, 14, 21, 28, ...
   d) 1, 2, 4, 7, 11, ...
   e) 20, 16, 12, 8, ...
   f) 32, 16, 8, 4, ...
   g) \( \frac{11}{3}, \frac{10}{3}, \frac{8}{3}, \frac{5}{3}, \ldots \)
   h) 0.5, 1.5, 4.5, 13.5, ...

2. State the common ratio and write the next three terms of each geometric sequence.
   a) 1, 3, 9, 27, ...
   b) 5, 10, 20, 40, ...
   c) 2, –8, 32, –128, ...
   d) 7, –7, 7, –7, ...
   e) 0.5, 5, 50, 500, ...
   f) \( \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots \)
   g) 64, 32, 16, 8, ...
   h) 800, –400, 200, –100, ...

3. Write the first 5 terms of each geometric sequence.
   a) \( a = 4 \) and \( r = 3 \)
   b) \( a = 20 \) and \( r = 4 \)
   c) \( a = 1024 \) and \( r = 0.5 \)
   d) \( a = 0.043 \) and \( r = 10 \)
   e) \( a = 8 \) and \( r = -1 \)
   f) \( a = -10 \) and \( r = -5 \)
4. Given the formula for the nth term of a geometric sequence, write the first 4 terms.
   a) \( t_n = 4(2)^{n-1} \)
   b) \( t_n = 10(3)^{n-1} \)
   c) \( t_n = 2(-2)^{n-1} \)
   d) \( f(n) = 5(-3)^{n-1} \)
   e) \( t_n = -3(2)^{n-1} \)
   f) \( t_n = -2(-3)^{n-1} \)
   g) \( f(n) = 0.5(4)^{n-1} \)
   h) \( t_n = -(1)^{n-1} \)
   i) \( f(n) = 200(0.5)^{n-1} \)
   j) \( f(n) = -1000(-0.1)^{n-1} \)

5. Find the formula for the nth term and find the indicated terms for each of the following geometric sequences.
   a) 2, 4, 8, … ; \( t_7 \) and \( t_{12} \)
   b) 1, 5, 25, … ; \( t_6 \) and \( t_9 \)
   c) 4, 12, 36, … ; \( t_8 \) and \( t_{10} \)
   d) 64, 32, 16, … ; \( t_7 \) and \( t_{10} \)
   e) 6, 0.6, 0.06, … ; \( t_6 \) and \( t_9 \)
   f) -3, 6, -12, … ; \( t_7 \) and \( t_9 \)
   g) 729, -243, 81, … ; \( t_8 \) and \( t_{10} \)
   h) 4, -40, 400, … ; \( t_9 \) and \( t_{12} \)

6. Find the number of terms in each of the following geometric sequences.
   a) 4, 12, 36, … , 2916
   b) 3, 6, 12, … , 1536
   c) 2, -4, 8, … , -1024
   d) 4374, 1458, 486, … , 2
   e) \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{1024} \)
   f) \( \frac{1}{25}, \frac{1}{5}, 1, \ldots, 625 \)
   g) \( \frac{1}{81}, \frac{4}{27}, \frac{8}{9}, \ldots, 6912 \)
   h) -409.6, 102.4, -25.6, … , 0.025

7. Given two terms of each geometric sequence, find \( t_n \) for the sequence.
   a) \( t_3 = 36, t_4 = 108 \)
   b) \( t_2 = 6, t_3 = -12 \)
   c) \( t_4 = 64, t_5 = 32 \)
   d) \( t_2 = 4, t_4 = 64 \)
   e) \( t_5 = 80, t_6 = 320 \)
   f) \( t_3 = 99, t_5 = 11 \)

Apply, Solve, Communicate

8. Measurement  The diagrams show the side lengths of three 30°-60°-90° triangles. Find the side lengths of the next triangle in the sequence.

9. Helium balloon  A balloon filled with helium has a volume of 20 000 cm³. The balloon loses one fifth of its helium every 24 h.
   a) Write the sequence giving the volume of helium in the balloon at the beginning of each day for 5 days, including the first day.
   b) What is the common ratio for this sequence?
   c) What volume of helium will be in the balloon at the start of the sixth day? the seventh day?
10. **Demographics** Each year for 10 years, the population of a city increased by 5% of its value in the previous year. If the initial population was 200,000, what was the population after 10 years?

11. **Vacuum pump** Each stroke of a vacuum pump removes one third of the air remaining in a container. What percent of the original quantity of air remains in the container after 10 strokes, to the nearest percent?

12. **Photocopying** Many photocopiers can reduce the dimensions of the image of an original. Usually the maximum reduction capability is to 64% of the original dimensions. How many reductions, at the maximum setting, would it take to reduce an image to less than 10% of its original dimensions?

13. **Biology** A single bacterium divides into two bacteria every 10 min. If the same rate of division continues for 2 h, how many bacteria will there be?

14. The first two terms of a sequence are 3 and 6.
   a) Write the first 5 terms of the sequence if the sequence is arithmetic.
   b) Write the first 5 terms of the sequence if the sequence is geometric.
   c) Graph each sequence and compare the graphs.

15. Which term of the geometric sequence 4, 12, 36, … , is 2916?

16. **Application** The rate of decay for a radioactive isotope varies from one isotope to another. The time it takes for half of any sample to decay is called the half-life. Barium-123 has a half-life of 2 min.
   a) Write an equation to determine the quantity of barium-123 remaining in a sample after $n$ half-lives.
   b) A fresh sample of 80 mg of barium-123 was obtained for an experiment. It took 10 min to set up the experiment. What mass of the barium-123 was left when the experiment began?

17. **Communication** If each term of a geometric sequence is multiplied by the same number, is the resulting sequence a geometric sequence? Explain.

18. **Rare stamp** The value of a rare stamp is expected to follow a geometric sequence from year to year. The stamp is now worth $800 and is expected to be worth $1250 two years from now. How much is it expected to be worth
   a) one year from now? 
   b) three years from now?

19. a) Is the graph of $t_n$ versus $n$ for a geometric sequence linear or non-linear?
   b) Use finite differences to explain your answer to part a).
20. **Motion of a pendulum** On the first swing, a pendulum swings through an arc of 50 cm. On each successive swing, the length of the arc is 0.97 of the previous length. What is the length of the arc, to the nearest hundredth of a centimetre, on

a) the 10th swing?  

b) the 15th swing?

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21. **Algebra** Show that \( t_n ÷ t_{n-1} = r \) for any geometric sequence.

22. **Algebra** Determine the value of \( x \) that makes each sequence geometric.

a) 4, 8, 16, 3\( x + 2 \), ...  
b) 2, 6, 5\( x - 2 \), ...

23. **Inquiry/Problem Solving** Can the side lengths of a triangle be consecutive terms of a geometric sequence? Explain your reasoning.

24. **Algebra** The first three terms of a geometric sequence are \( w, x, \) and \( y \). Express \( y \) in terms of \( w \) and \( x \). Check your solution using the first three terms of any geometric sequence.

25. **Geometric mean** If \( a, x, \) and \( b \) are consecutive terms of a geometric sequence, then \( x \) is called the geometric mean of \( a \) and \( b \). Find the geometric mean of

a) 2 and 8  
b) 5 and 180  
c) \( m \) and \( n \)

26. Find \( a, r, \) and \( t_n \) for the following geometric sequences.

a) \( t_5 = 48 \) and \( t_8 = 384 \)  
b) \( t_3 = 24 \) and \( t_6 = -192 \)

27. **Algebra** Find \( a, r, \) and \( f(n) \) for the following geometric sequences.

a) \( f(3) = 5x^6 \) and \( f(10) = 5x^{20} \)  
b) \( f(4) = 8x^3 \) and \( f(9) = 256x^8 \)

28. **Algebra** Find the indicated terms for each of the following geometric sequences.

a) \( t_n \) and \( t_{10} \) for \( 2x, 4x^2, 8x^3, ... \)  
b) \( t_n \) and \( t_6 \) for \( \frac{1}{2}, \frac{x}{4}, \frac{x^2}{8}, ... \)

a) \( t_n \) and \( t_{25} \) for \( \frac{1}{x^4}, \frac{1}{x^2}, 1, ... \)  
b) \( t_n \) and \( t_{20} \) for \( 3x^{10}, -3x^9, 3x^8, ... \)
CAREER CONNECTION  Accounting

All organizations need to monitor and record their financial activities. Accountants are the professionals who measure and report on the financial activities of organizations. The history of accounting can be traced back to ancient times. The first official meeting of accountants in North America was held in Montréal in 1879.

Accounting reports are used within an organization to plan for the future. They are also made available to selected outside groups, such as banks and government tax departments, that need to be aware of the organization’s finances. These outside groups employ their own accountants to interpret the reports they receive and to monitor their own financial affairs.

1. **Depreciation**  The Canada Customs and Revenue Agency administers Canada’s tax laws. The agency allows tax deductions based on the depreciation of equipment used for business purposes. The agency allows a 30% annual depreciation rate for computer equipment. Suppose that you are an accountant working for a company that purchased $60,000 worth of computer equipment today.

   a) Calculate the value that the Canada Customs and Revenue Agency will assign to the computer equipment four years from now.

   b) In the company’s tax return for any year, you can claim a tax deduction equal to the decrease in value of the computer equipment that year. Calculate the deduction you will claim for the fourth year from now.

2. **Research**  Use your research skills to determine the meanings of the accounting qualifications CA, CMA, and CGA. Compare the education and training required for these qualifications, and describe some possible career paths open to people who obtain them.