Solutions

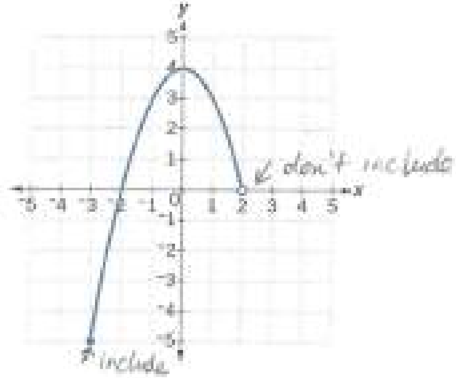
Unit 1: Functions

1. State the domain and range.

$D=\{x\in\mathbb{R} \mid -3\le x<2\}$

Is this a function? $R=\{y\in\mathbb{R} \mid -5\le y\le 4\}$

Yes, it passes the vertical line test.



2. Given that $f(x) = x^2 + 7$ and $g(x) = -3x + 1$:

a) Evaluate $f(-2)$. $= 11$

b) Simplify $4g(b + c)$. $= -12b - 12c + 4$

c) Simplify $f(g(x))$. $= 9x^2 - 6x + 8$

d) Solve for x if $f(x) = 23$. $x = \pm 4$

3. For the quadratic function $f(x) = -5x^2 + 15x - 9$, determine the min/max value and when it occurs, by:

a) Completing the square

b) Partial factoring

max. value $y = \frac{9}{4}$ when $x = \frac{3}{2}$

4. Simplify.

a) $\sqrt{48} - \sqrt{27} + \sqrt{12}$ $= 3\sqrt{3}$

b) $5\sqrt{3} \times 3\sqrt{2}$ $= 15\sqrt{6}$

c) $\frac{15\sqrt{48}}{5\sqrt{3}}$ $= 12$

d) $\frac{3}{\sqrt{3}-4}$ $= \frac{-3\sqrt{3}-12}{13}$

5. Factor fully: $2x^2 + 6x^3 + 5x^7 + 15x^8$.

$= x^2(3x+1)(5x^5+2)$

6. Solve:

a) $6x^2 + 5x = 4$ Factor.

$x = \frac{1}{2}$ or $x = \frac{-4}{3}$

b) $4x^2 - 10x + 5 = 0$ Quad. Form.

$x = \frac{5 \pm \sqrt{5}}{4}$

7. Algebraically determine the equation of a parabola which passes through the point $(1, -1)$ with roots at $2 \pm \sqrt{3}$.

$y = \frac{1}{2}(x-(2+\sqrt{3}))(x-(2-\sqrt{3}))$

8. The sum of the squares of two consecutive odd numbers is 394. Algebraically determine the numbers.

The numbers are 13 and 15 or -13 and -15.

9. Solve the following linear quadratic system:

① $2y + 6 = x$ ② $y^2 - 9 = 0$

Solutions are:
$(0,-3)$ and $(12,3)$.

Unit 2

1. Simplify. State any restrictions.

a) $\frac{x^2-16}{x^2-x-12}$ $= \frac{x+4}{x+3}$, $x \neq -3, 4$

b) $\frac{2x^2-x-1}{3x^2+x-2} \div \frac{2x^2-3x-2}{3x^2-11x+6}$

$= \frac{(x-1)(x-3)}{(x+1)(x-2)}$, $x \neq -1, -\frac{1}{2}, \frac{2}{3}, 2, 3$

c) $\frac{x+1}{3x^2+4x+1} + \frac{2x-1}{3x^2-5x-2}$

$= \frac{3(x-1)}{(3x+1)(x-2)}$, $x \neq -1, -\frac{1}{3}, 2$

2. Describe the transformations of the following function from the graph of $f(x)$: in order

$$g(x) = -2f\left(\frac{1}{3}(x+3)\right) - 6$$

① V.S. by a factor of 2.
② H.S by a factor of 3.
③ Reflection in the x-axis.
④ H.T. 3 units left, V.T. 6 units down.

3. Given $f(x) = x^2 + 6x$,

a) Write equations for $-f(x)$ and $f(-x)$.

b) Determine any points that are invariant.

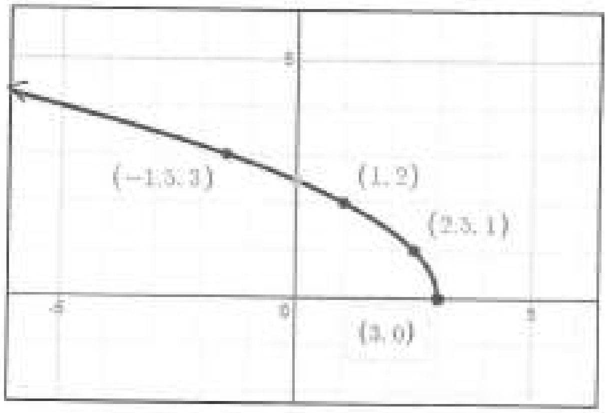
a) $-f(x) = -x^2 - 6x$ $\quad f(-x) = x^2 - 6x$

b) invariant pts. lie on the x-axis, ∴ the zeros. $(0,0)$ & $(-6,0)$

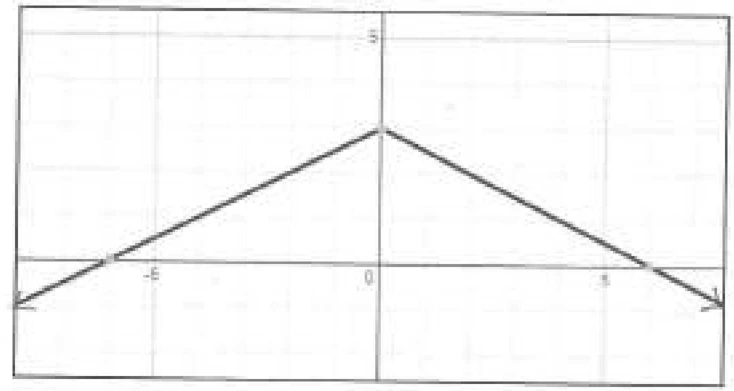
invariant pts. lie on the y-axis, ∴ the y-int. $(0,0)$

4. Sketch the following graphs showing key points:

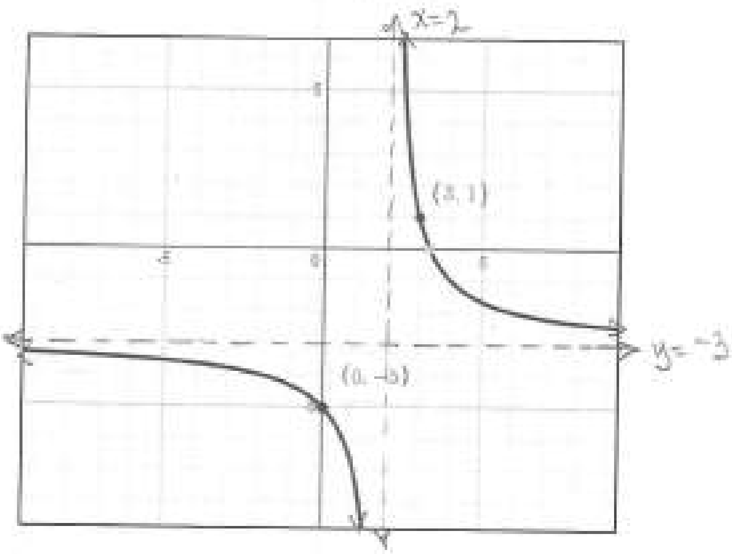
a) $y = \sqrt{-2x+6}$



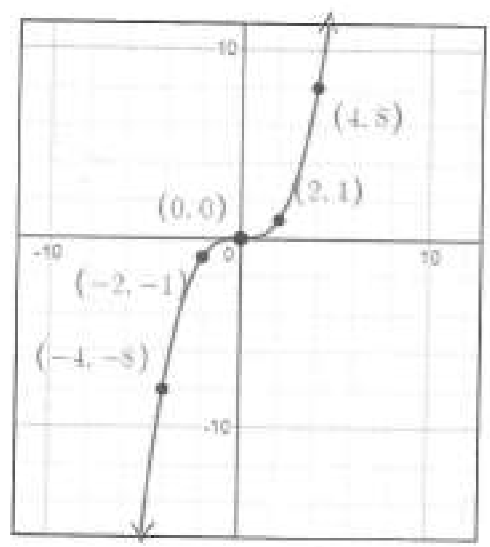
b) $y = -\frac{1}{2}|x| + 3$



c) $f(x) = \frac{4}{x-2} - 3$



d) $f(x) = \left(\frac{1}{2}x\right)^3$



5. Algebraically determine the inverse of the function $f(x) = 2(x+4)^2 - 1$. Is the inverse a function?

$f^{-1}(x) = -4 \pm \sqrt{\frac{x+1}{2}}$

2 branches ∴ will fail the vertical line test. Not a function.

Unit 3

1. Simplify. No decimals may be used.

a) $\left(\frac{4x^{-3}y^{4}}{8x^{2}y^{-2}}\right)^{-2}$

$= \frac{4x^{10}}{y^{12}}$

b) $\left(\frac{27}{125}\right)^{-\frac{2}{3}}$

$= \frac{25}{9}$

c) $\frac{\left(p^{\frac{-3}{4}}q^{3}\right)^{\frac{1}{3}}}{(p^{-2}q^{4})(p^{2}q^{4})^{\frac{1}{2}}}$

$= \frac{p^{-\frac{1}{4}}q}{(p^{-2}q^{4})(pq^{2})}$

$= \frac{p^{-\frac{1}{4}}q}{p^{-1}q^{6}}$

$= \frac{p^{3/4}}{q^{5}}$

d) $\left(\sqrt[4]{\sqrt[3]{\sqrt{64}}}\right)^{8}$

$= \left(\sqrt[4]{\sqrt[3]{8}}\right)^{8}$

$= \left(\sqrt[4]{2}\right)^{8}$

$= \left(2^{\frac{1}{4}}\right)^{8}$

$= 2^{2} = 4$

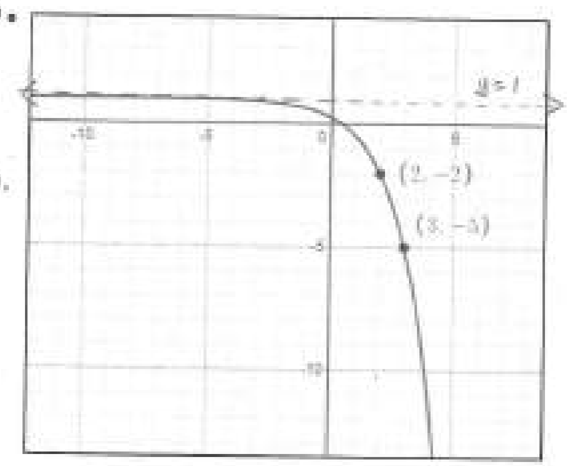
2. Given the function $y = -3(2)^{x-2} + 1$,

a) Describe the transformations.

b) Graph the function.

$y = 2^{x}$ is the base function.

① V.S. by a factor of 3.
② Reflection in the x-axis.
③ H.T. right 2, V.T. up 1.



c) State the y-intercept, the domain, the range and the equation of the asymptote.

$D = \{x \in \mathbb{R}\}$ $R = \{y \in \mathbb{R} \mid y < 1\}$

y-int: set $x=0$, solve. $y = 1$

$y = -3(2)^{-2} + 1$

$y = -3\left(\frac{1}{4}\right) + 1$

$y = -\frac{3}{4} + \frac{4}{4}$ $\therefore \left(0, \frac{1}{4}\right)$ is the y-int.

$y = \frac{1}{4}$

3. Solve the following exponential equations:

a) $5(4^x) = 10$

$4^x = \frac{10}{5}$

$4^x = 2$

$(2^2)^x = 2$

$2^{2x} = 2^1$

$\because$ bae, $2x = 1$ and $x = \frac{1}{2}$

b) $16^{2p+1} = 8^{4p+5}$

$(2^4)^{2p+1} = (2^3)^{4p+5}$

$2^{8p+4} = 2^{12p+15}$

$\because$ bae, $8p + 4 = 12p + 15$

$-11 = 4p$

$-\frac{11}{4} = p$

c) $3^{3x-1} = \frac{1}{81}$

$3^{3x-1} = 3^{-4}$

$\because$ bae,

$\therefore 3x - 1 = -4$

$3x = -3$

$x = -1$

d) $2^{x+5} + 2^x = 1056$

$2^x \cdot 2^5 + 2^x = 1056$

$2^x(2^5 + 1) = 1056$

$2^x(33) = 1056$

$2^x = 32$

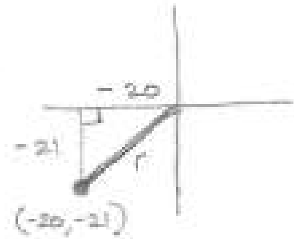
$\therefore x = 5$

4. The half-life of Vanadium-48 is 32 hours. How long will it take an initial amount of 368 g to decay to 23 g?

It will take 128 hours.

Unit 4

1. The point $(-20, -21)$ is on the terminal arm of an angle θ in standard position. Find $\sin\theta$ and $\cos\theta$.



$r^2 = (-20)^2 + (-21)^2$
$r = 29$

$\sin\theta = \frac{y}{r} = \frac{-21}{29}$

$\cos\theta = \frac{x}{r} = \frac{-20}{29}$

2. Evaluate exactly:

a) sin 330°

$= -\sin 30°$
$= -\frac{1}{2}$

b) sec 120°

$= -\sec 60°$
$= -2$

c) tan 270°

= undefined.

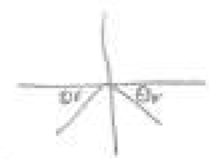
3. Determine all values of θ, where $0° \leq \theta \leq 360°$.

a) $\cos\theta = -\frac{1}{\sqrt{2}}$

$\theta_r = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$
$\theta_r = 45°$
Q2: $\theta = 180° - 45° = 135°$
Q3: $\theta = 180 + 45° = 225°$
$\therefore \theta = \{135°, 225°\}$

b) $\sin\theta = -0.1573$

$\theta_r = \sin^{-1}(0.1573)$
$\theta_r \doteq 9°$
Q3: $\theta = 180° + 9° = 189°$
Q4: $\theta = 360° - 9° = 351°$
$\therefore \theta = \{189°, 351°\}$

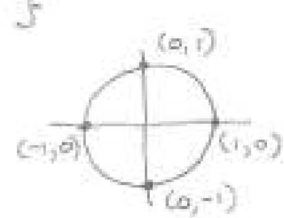


c) $\cot\theta = \frac{1}{\sqrt{3}}$

$\tan\theta = \sqrt{3}$
$\theta_r = \tan^{-1}(\sqrt{3})$
$\theta_r = 60°$
Q3: $\theta = 180° + 60° = 240°$
$\therefore \theta = \{60°, 240°\}$

d) $\csc\theta = -1$

$\theta = 270°$



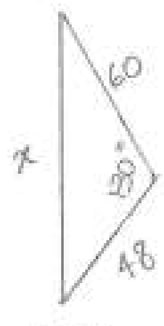
$\csc\theta = \frac{r}{y}$
$\csc\theta = \frac{1}{-1}$
where is y = -1?

4. Two ships left Port Hope on Lake Ontario at the same time. One travelled at 12 km/h on a course of 235°. The other travelled at 15 km/h on a course of 105°. How far apart were the ships after four hours, to the nearest kilometer?

$d = vt$

$d_1 = 12(4) = 48$ km

$d_2 = 15(4) = 60$ km



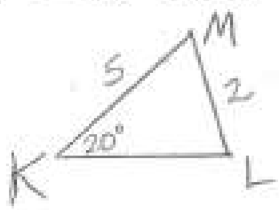
SAS
∴ Cosine Law

$x^2 = 48^2 + 60^2 - 2(48)(60)\cos 130°$

$x \doteq 98$

∴ The ships were 98 km apart.

5. Solve ΔKLM, given $< K = 20°, k = 2 \quad cm$ and $l = 5\ cm.$



Given SSA, must consider ambiguous case.

① Solve for ∠L.

$\frac{\sin L_1}{5} = \frac{\sin 20°}{2}$

$\angle L_1 = \sin^{-1}\left(\frac{5\sin 20°}{2}\right)$

$\angle L_1 \doteq 59°$

① In Q2:

$\angle L_2 = 180° - 59°$

$\angle L_2 \doteq 121°$

② Find ∠M using ASTT.

$\angle M_1 = 180° - 20° - 59°$

$\angle M_1 = 101°$

②

$\angle M_2 = 180° - 20° - 121°$

$\angle M_2 = 39°$

∴ There are 2 Δ's.

Solutions

$\angle L_1 = 59°$ or $\angle L_2 = 121°$

$\angle M_1 = 101°$ $\quad \angle M_2 = 39°$

$m_1 = 5.7$ cm $\quad m_2 = 3.7$ cm

③ Solve for m.

$\frac{m_1}{\sin 101°} = \frac{2}{\sin 20°}$

$m_1 = \frac{2\sin 101°}{\sin 20°}$

$m_1 \doteq 5.7$

③

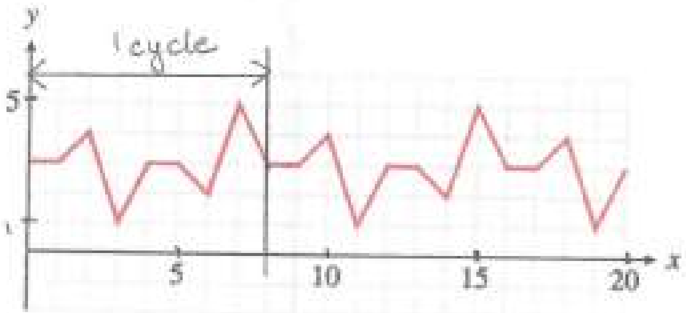
$\frac{m_2}{\sin 39°} = \frac{2}{\sin 20°}$

$m_2 = \frac{2\sin 39°}{\sin 20°}$

$m_2 \doteq 3.7$

Unit 5

1. For the periodic function shown, state the period, the amplitude and the equation of the midline.



per = 8

max = 5
min = 1

amp = $\frac{max - min}{2}$
$= \frac{5-1}{2}$
$= 2$

c = $\frac{max + min}{2}$
$= \frac{5+1}{2}$
$= 3$

∴ The period is 8, the amplitude is 2 and the equation of the midline is $y = 3$.

2. Given $y = 3\sin 3(\theta + 45°) + 1$, state the amplitude, period and phase shift. Graph one cycle of the function.

amp = 3

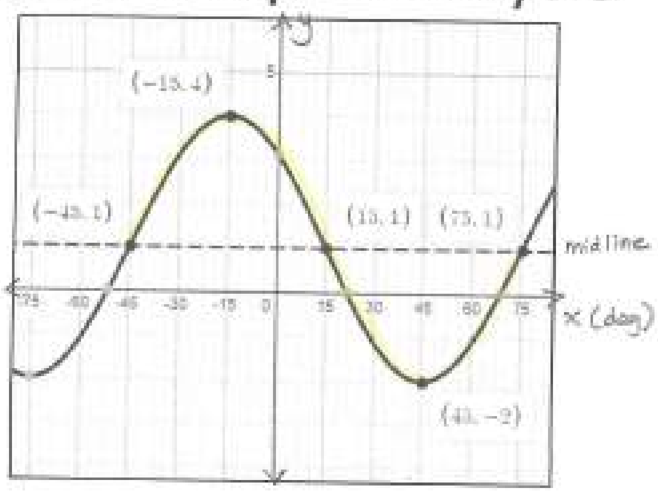
per = $\frac{360°}{K}$
$= \frac{360°}{3}$
$= 120°$

p.s. = 45° left

On graph notice:
max = 1 + 3 = 4
min = 1 − 3 = −2

start at x = −45°,
end at x = −45° + 120°
= 75°
(on midline, ∵ sine).

spacing = $\frac{120°}{4}$ = 30° (∴ scale is by 15°).



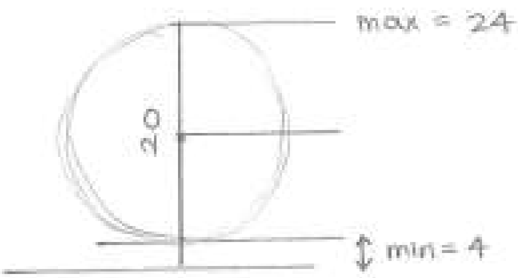
3. Write the equation of the cosine function that has an amplitude of 3, a period of 720°, a phase shift right 30° and a vertical translation down 2 units.

Given
a = 3
d = 30°
c = −2

per = 720°
K = $\frac{360°}{per}$
K = $\frac{360°}{720°}$
K = $\frac{1}{2}$

∴ The eq'n is $y = 3\cos\left[\frac{1}{2}(x - 30°)\right] - 2$.

4. A Ferris wheel has a diameter of 20 m and is 4 m above ground level at its lowest point. The Ferris wheel completes one revolution in 120 seconds. Assuming that a rider enters a car from a platform that is located at the bottom of the Ferris wheel,



$c = \frac{24+4}{2}$

$c = 14$

$a = \frac{24-4}{2}$

$a = 10$

the radius

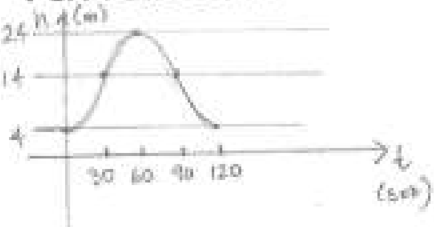
per = 120 s

$k = \frac{360^\circ}{120\text{ s}}$

$k = 3^\circ/\text{s}$

a) Model the rider's height above the ground versus ~~the angle of rotation~~ time, using a transformed sine function.

Let t be the amount of time in seconds.
Let $h(t)$ be the rider's height above the ground in m.



$$h(t) = 10 \sin[3(t-30)] + 14$$

b) Repeat part a) with a transformed cosine function.

$$h(t) = -10 \cos 3t + 14$$

or $$h(t) = 10 \cos[3(t-60)] + 14$$

Unit 6

1. Determine the simplified formula for the n^{th} term and the indicated term for each sequence:

a) $-4, 3, 10, \ldots; t_{18}$

$a = -4$
$d = 3-(-4)$
$= 7$

① $t_n = a + (n-1)d$
$t_n = -4 + (n-1)7$
$t_n = -4 + 7n - 7$
$\boxed{t_n = 7n - 11}$

② $t_{18} = 7(18) - 11$
$\boxed{t_{18} = 115}$

b) $1, -3, 9, \ldots; t_7$

$a = 1$
$r = \frac{-3}{1}$
$r = -3$

① $t_n = ar^{n-1}$
$t_n = 1(-3)^{n-1}$
$\boxed{t_n = (-3)^{n-1}}$

② $t_7 = (-3)^{7-1}$
$t_7 = (-3)^6$
$\boxed{t_7 = 729}$

2. Algebraically determine the number of terms for each sequence:

a) $19, 11, 3, \ldots, -229$

$a = 19$
$d = 11 - 19$
$d = -8$
$n = ?$
$t_n = -229$

$t_n = a + (n-1)d$
$-229 = 19 + (n-1)(-8)$
$-248 = -8(n-1)$
$31 = n - 1$
$32 = n$

∴ There are 32 terms.

b) $27, 9, 3, \ldots, \frac{1}{2187}$

$a = 27$
$r = \frac{9}{27}$
$r = \frac{1}{3}$
$t_n = \frac{1}{2187}$
$n = ?$

$t_n = ar^{n-1}$
$\frac{1}{2187} = 27\left(\frac{1}{3}\right)^{n-1}$
$\frac{1}{59049} = \left(\frac{1}{3}\right)^{n-1}$
$\left(\frac{1}{3}\right)^{10} = \left(\frac{1}{3}\right)^{n-1}$

∵ bases are equal,
∴ $10 = n - 1$
$11 = n$

∴ There are 11 terms.

3. Write the first 5 terms of the sequence given the following recursion formula:

$$t_1 = 3; t_2 = 4; t_n = t_{n-1} + t_{n-2}$$

$t_3 = t_2 + t_1$
$= 4 + 3$
$= 7$

$t_4 = t_3 + t_2$
$= 7 + 4$
$= 11$

$t_5 = t_4 + t_3$
$= 11 + 7$
$= 18$

∴ The first five terms are 3, 4, 7, 11, 18.

4. Find the sum of each series.

a) $1 + \frac{5}{4} + \frac{3}{2} + \cdots + 20$

$d_1 = \frac{5}{4} - \frac{4}{4} = \frac{1}{4}$

$d_2 = \frac{3}{2} - \frac{5}{4}$

$= \frac{6}{4} - \frac{5}{4}$

$= \frac{1}{4}$

$\therefore$ arithmetic

$a = 1$

$t_n = 20$

$n = ?$

$S_n = ?$

① $t_n = a + (n-1)d$

$20 = 1 + (n-1)\left(\frac{1}{4}\right)$

$19 = \frac{1}{4}(n-1)$

$76 = n - 1$

$77 = n$

② $S_n = \frac{n}{2}[a + t_n]$

$S_{77} = \frac{77}{2}[1 + 20]$

$S_{77} = \frac{1617}{2}$

b) $3645 - 1215 + 405 - \cdots + 5$

$a = 3645$

$r = \frac{-1215}{3645}$

$r = \frac{-1}{3}$

$t_n = 5$

$n = ?$

$S_n = ?$

① $t_n = ar^{n-1}$

$5 = 3645\left(\frac{-1}{3}\right)^{n-1}$

$\frac{5}{3645} = \left(\frac{-1}{3}\right)^{n-1}$

$\frac{1}{729} = \left(\frac{-1}{3}\right)^{n-1}$

$\left(\frac{-1}{3}\right)^{6} = \left(\frac{-1}{3}\right)^{n-1}$

$\because$ base are equal,

$\therefore 6 = n - 1$

$7 = n$

② $S_n = \frac{a(r^n - 1)}{r - 1}$

$S_7 = \frac{3645\left[\left(\frac{-1}{3}\right)^7 - 1\right]}{\frac{-1}{3} - \frac{3}{3}}$

$= \frac{3645\left(\frac{-1}{2187} - \frac{2187}{2187}\right)}{-\frac{4}{3}}$

$= 3645\left(\frac{-2188}{2187}\right)\left(\frac{-3}{4}\right)$

$S_7 = 2735$

5. The Women's World Cup of Soccer tournament was first held in 1991. The next two tournaments were held in 1995 and 1999. Assuming that this pattern continues to repeat, algebraically determine the year of the 35th tournament.

t_1 1st tourn, t_2 2nd tour, t_3 3rd tour

The sequence is: 1991, 1995, 1999, ..., t_{35}

$a = 1991$

$d = 4$

$n = 35$

$t_{35} = ?$

$t_n = a + (n-1)d$

$t_{35} = 1991 + (35-1)(4)$

$= 1991 + 34(4)$

$= 2127$

$\therefore$ The 35th tournament will be held in 2127.

6. One day you saw an awesome video on YouTube. At 1 pm, you shared a video link to 5 unique people. At 2 pm, each of your friends shared it to 5 unique people. At 3 pm, each of their friends shared it to 5 unique people. If this pattern keeps repeating, algebraically determine how many unique people have received the link by 11 pm.

The series is:

$$\underset{t_1}{\overset{1\text{pm}}{5}} + \underset{t_2}{\overset{2\text{pm}}{25}} + \underset{t_3}{\overset{3\text{pm}}{125}} + \ldots + \overset{11\text{pm}}{t_{11}}$$

$a = 5$

$r = \frac{25}{5}$ ← each 5 friends share with 5 others, so 5x5 = 25 geometric

$r = 5$

$n = 11$

$S_{11} = ?$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{11} = \frac{5(5^{11} - 1)}{5 - 1}$$

$$S_{11} = 61\ 035\ 155$$

∴ 61 035 155 unique people will have received the video!

7. Determine the missing portions of this section of Pascal's Triangle:

$$\begin{array}{r} 120 \\ +210 \\ \hline 330 \end{array} \qquad \begin{array}{r} 330 \\ +462 \\ \hline 792 \end{array} \qquad \begin{array}{r} 495 \\ -330 \\ \hline 165 \end{array}$$

8. How many downward paths can be taken to spell MATHEMATICS?

M

$A_1 \quad A$

$T_1 \quad T_2 \quad T$

$H_1 \quad H_3 \quad H_3 \quad H$

$E_1 \quad E_4 \quad E_6 \quad E_4 \quad E$

$M_1 \quad M_5 \quad M_{10} \quad M_{10} \quad M_5 \quad M$

$A_6 \quad A_{15} \quad A_{20} \quad A_{15} \quad A_6$

$T_{21} \quad T_{35} \quad T_{35} \quad T_{21}$

$I_{56} \quad I_{70} \quad I_{56}$

$C_{126} \quad C_{126}$

S_{252}

$\therefore$ There are 252 downward paths.

9. Expand and simplify $\left(a - \frac{b}{2}\right)^5$.

$$= \binom{5}{0}(a^5)\left(\frac{-b}{2}\right)^0 + \binom{5}{1}(a^4)\left(\frac{-b}{2}\right)^1 + \binom{5}{2}(a^3)\left(\frac{-b}{2}\right)^2 + \binom{5}{3}(a^2)\left(\frac{-b}{2}\right)^3 + \binom{5}{4}(a^1)\left(\frac{-b}{2}\right)^4 + \dots$$

$$\dots + \binom{5}{5}(a^0)\left(\frac{-b}{2}\right)^5$$

$$= a^5 + 5a^4\left(\frac{-b}{2}\right) + 10a^3\left(\frac{b^2}{4}\right) + 10a^2\left(\frac{-b^3}{8}\right) + 5a\left(\frac{b^4}{16}\right) - \frac{b^5}{32}$$

$$= a^5 - \frac{5}{2}a^4b + \frac{5}{2}a^3b^2 - \frac{5}{4}a^2b^3 + \frac{5}{16}ab^4 - \frac{1}{32}b^5$$

Unit 7

1. Kadeem invested in a GIC that paid 3.25% simple interest. In 36 months, he earned \$485. How much did he invest originally?

$r = 0.0325$
$t = \frac{36}{12} = 3$ years
$A = 485$
$P = ?$

$I = Prt$
$A = P + I$
$A = P + Prt$
$A = P(1+rt)$
$P = \frac{A}{1+rt}$

$P = \frac{485}{1+0.0325(3)}$
$P = \frac{485}{1.0975}$
$P \doteq 441.91$

∴ Kadeem invested \$441.91 originally.

2. Determine the amount of a \$2200 investment, compounded monthly for 5 years at 12% per annum.

$P = 2200$
$C = 12$
$t = 5$
$r = 0.12$
$i = \frac{0.12}{12}$
$n = 5(12)$

$A = P(1+i)^n$
$= 2200\left(1+\frac{0.12}{12}\right)^{5(12)}$
$\doteq \$3996.73$

3. Faris needs \$5000 for university in 3 years. His parents invest some money in an account paying interest at a rate of 7.1% per annum, compounded quarterly. How much should they invest now to have \$5000 in 3 years?

$FV = 5000$
$t = 3$
$r = 0.071$
$C = 4$
$PV = ?$
$i = \frac{0.071}{4}$
$n = 3(4)$

$P = A(1+i)^{-n}$
$= 5000\left(1+\frac{0.071}{4}\right)^{-12}$
$\doteq \$4048.34$

4. Marianna deposited \$200 into her bank account at the end of each month for 8 months. The account pays 2.9% per annum, compounded monthly. How much is in her account at the end of the 8 months?

$R = 200$
$n = 8$
$r = 0.029$
$C = 12$
$A = ?$
$i = \frac{0.029}{12}$

$$A = \frac{R[(1+i)^n - 1]}{i}$$

$$= \frac{200\left[\left(1+\frac{0.029}{12}\right)^8 - 1\right]}{\frac{0.029}{12}}$$

$$\doteq \$1613.60$$

5. Mrs. Stewart bought a new car. She financed \$13 500 at 3.9% compounded monthly and chose to make monthly payments for 4 years.

a) What amount does Mrs. Stewart pay per month?

$PV = 13\,500$
$r = 0.039$
$C = 12$
$t = 4$
$i = \frac{0.039}{12}$
$n = 12(4)$
$R = ?$

$$R = \frac{Pi}{[1-(1+i)^{-n}]}$$

$$= \frac{13500\left(\frac{0.039}{12}\right)}{\left[1-\left(1+\frac{0.039}{12}\right)^{-48}\right]}$$

$$\doteq 304.21$$

b) How much interest did Mrs. Stewart pay to the dealership in order to finance her new car?

$$I = 48(304.21) - 13\,500$$
$$= \$1102.08$$