5.2 The Binomial Theorem

$a + b$ binomial - an expression with two terms

Consider:

$(a + b)^0 = 1$

$(a + b)^1 = 1a + 1b$

$(a + b)^2 = 1a^2 + 2ab + 1b^2$

Can you sense any familiar patterns?

$$(a + b)^3 = (a + b)(a^2 + 2ab + b^2)$$
$$= 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a + b)^4 = (a + b)(a^3 + 3a^2b + 3ab^2 + b^3)$$
$$= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

$$(a + b)^5 = (a + b)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)$$
$$= 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

How many terms? $n=2$, 3 terms
$n=4$, 5 terms

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a^1 + 1b^1$$

$$(a+b)^2 = 1a^2 + 2a^1b^1 + 1b^2$$

$$(a+b)^3 = 1a^3 + 3a^2b^1 + 3a^1b^2 + 1b^3$$

$$(a+b)^4 = 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$$

$$(a+b)^5 = 1a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1b^5$$

The coefficients of each term in the binomial expansion of $(a+b)^n$ correspond with the terms in row n of Pascal's Triangle!

The Binomial Theorem:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

or,

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

Sum of terms from $r=0$ to $r=n$

The coefficients of the form $\binom{n}{r}$ are called *binomial* coefficients.

Example 2: Use the binomial theorem to expand the following.

a. $(x+y)^4$

$$=\binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}x^0y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

b. $(4x+y)^4$

$$=\binom{4}{0}(4x)^4y^0 + \binom{4}{1}(4x)^3y^1 + \binom{4}{2}(4x)^2y^2 + \binom{4}{3}(4x)^1y^3 + \binom{4}{4}(4x)^0y^4$$

$$= 1\cdot 4^4x^4\cdot 1 + 4\cdot 4^3x^3y + 6\cdot 4^2x^2y^2 + 4\cdot 4\cdot xy^3 + 1\cdot 1\cdot y^4$$

$$= 256x^4 + 256x^3y + 96x^2y^2 + 16xy^3 + y^4$$

c. $(x-2)^5$

$$=\binom{5}{0}x^5(-2)^0 + \binom{5}{1}x^4(-2)^1 + \binom{5}{2}x^3(-2)^2 + \binom{5}{3}x^2(-2)^3$$

$$+\binom{5}{4}x(-2)^4 + \binom{5}{5}x^0(-2)^5$$

$$= x^5 + 5x^4(-2) + 10x^3(4) + 10x^2(-8) + 5x(16) + (-32)$$

$$= x^2 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$$

Example 3: Expand and simplify $\left(x - \frac{1}{x^2}\right)^3$

$a = x$

$b = -\frac{1}{x^2}$

$n = 3$

$$= \binom{3}{0} x^3 \left(-\frac{1}{x^2}\right)^0 + \binom{3}{1} x^2 \left(-\frac{1}{x^2}\right)^1 + \binom{3}{2} x \left(-\frac{1}{x^2}\right)^2 + \binom{3}{3} x^0 \left(-\frac{1}{x^2}\right)^3$$

$$= x^3 + 3\cancel{x^2}\left(-\frac{1}{\cancel{x^2}}\right) + 3x\left(\frac{1}{x^4}\right) + \left(-\frac{1}{x^6}\right)$$

$$= x^3 - 3 + \frac{3}{x^3} - \frac{1}{x^6}$$

Example 4: Rewrite $1 + 12x^3 + 54x^6 + 108x^9 + 81x^{12}$ in the form $(a + b)^n$.

We know $a^4 = 1$ 5 terms, $\therefore n = 4$

$b^4 = 81x^{12}$

$a^4 = 1$

$a = 1$ or -1

$b^4 = 81x^{12}$

$= 3^4 (x^3)^4$

$= (3x^3)^4$

$b = 3x^3$ or $-3x^3$

Since signs are not alternating, we know either both positive, or both negative

$\therefore (a+b)^n = (1+3x^3)^4$

OR

$= (-1-3x^3)^4$

Practice

pg.10 # 3, 4, 5, 13,15

pg. 13 # 6,7, challenge