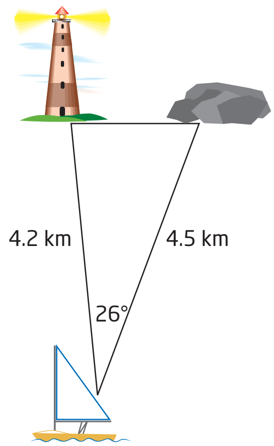
8.2 The Cosine Law

Sailing can be a lot of fun as long as you are careful to avoid hazardous situations, such as shallow water. Occasionally, when the ship's navigational instruments are out of order, a ship's captain can use trigonometry to find important distances and directions.

Suppose that as captain you know the measures of two sides and one angle in the acute triangle shown. Can you use the sine law to solve the triangle?



Tools

- protractor and ruler

OR

- computer with *The Geometer's Sketchpad®*

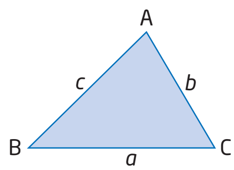
Technology Tip

If you are using *The Geometer's Sketchpad®*, calculate the square of a measure by clicking **^2** after it.

To calculate $2ab(\cos C)$, use the * key for multiplication and access the cosine function from the **Functions** drop-down menu on the calculator.

Investigate

How are the side lengths and cosines of angles related in an acute triangle?



1. a) Draw an acute △ABC.

b) Measure the side lengths a, b, and c.

c) Measure ∠C.

2. a) Calculate a^2, b^2, and c^2.

b) For acute △ABC, compare $a^2 + b^2$ and c^2. When would $a^2 + b^2 = c^2$? What is this relationship called?

c) Calculate $2ab(\cos C)$.

d) Describe any relationship you notice between $a^2 + b^2$, c^2, and $2ab(\cos C)$.

e) Examine two or three other acute triangles to see if this relationship holds true.

3. Suppose ∠C is 90°.

a) What happens to the quantity $2ab(\cos C)$?

b) What relationship applies to the three sides of the triangle when this happens?

4. **Reflect**

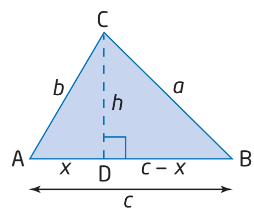
a) Write an equation that relates the cosine of an angle and the three sides of an acute triangle.

b) What happens to this relationship when the measure of one of the angles is 90°?

cosine law

- the relationship between the cosine of an angle and the lengths of the three sides in any acute △ABC:
$a^2 = b^2 + c^2 - 2bc(\cos A)$
$b^2 = c^2 + a^2 - 2ca(\cos B)$
$c^2 = a^2 + b^2 - 2ab(\cos C)$

The **cosine law** relates the cosine of an angle to the three side lengths of an acute triangle. To derive the cosine law, draw a triangle and add an altitude, h, from one of the vertices.



The altitude splits △ABC into two smaller right triangles, △ADC and △BDC.

Let AD = x.
Then, BD = $c - x$.

Focus on △ADC:
From the Pythagorean theorem, $b^2 = x^2 + h^2$.
Also, the cosine ratio gives

$$\frac{x}{b} = \cos A$$
$$x = b(\cos A)$$

Focus on △BDC:
Write an equation using the Pythagorean theorem.

$a^2 = h^2 + (c - x)^2$
$= h^2 + c^2 - 2cx + x^2$ — Expand the binomial.
$= x^2 + h^2 + c^2 - 2cx$ — Rearrange the terms.
$= b^2 + c^2 - 2c[b(\cos A)]$ — Substitute b^2 for $x^2 + h^2$ and $b(\cos A)$ for x.
$= b^2 + c^2 - 2cb(\cos A)$
$= b^2 + c^2 - 2bc(\cos A)$

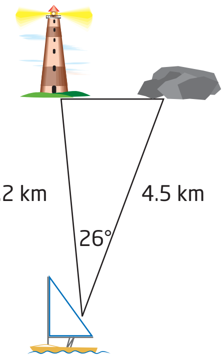
This equation allows you to find the side length a if you know the side lengths b and c and the measure of ∠A.

You can derive similar equations for the other side lengths. Combining the results gives the three forms of the cosine law.

$a^2 = b^2 + c^2 - 2bc(\cos A)$
$b^2 = c^2 + a^2 - 2ca(\cos B)$
$c^2 = a^2 + b^2 - 2ab(\cos C)$

Example 1 Find a Side Length Using the Cosine Law

A boat is sailing north through a narrow strait. Through one particularly narrow section, a lighthouse marks the western shoreline, while a buoy indicates a rock hazard directly east of the lighthouse, as shown. What is the width of this section of the strait, to the nearest tenth of a kilometre?



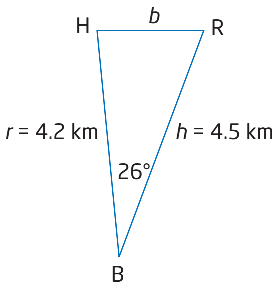
Did You Know?

The Clipper Round The World Yacht Race gives paying amateur crew members the chance to sail around the world. The race started in 1996 and takes place every 2 years.

It is the world's longest yacht race, at over 35 000 nautical miles, and takes the fleet of identical yachts 10 months to complete.

Solution

Draw a simplified diagram and label the given information. Choose variables for the vertices.



Since two sides and the contained angle are given, use the cosine law to find b.

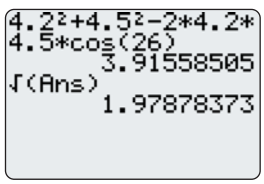
$b^2 = r^2 + h^2 - 2rh(\cos B)$

$b^2 = 4.2^2 + 4.5^2 - 2(4.2)(4.5)(\cos 26°)$ **Substitute the given information.**

$b^2 = 3.9155...$

$b = \sqrt{3.9155...}$ **Take the square root of both sides.**

$b \doteq 2.0$



The width of this section of the strait is approximately 2.0 km.

Literacy Connections

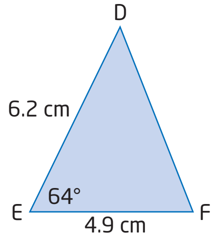
A contained angle is the interior angle that is formed at the vertex of two adjacent sides in a triangle.

Example 2 Solve a Triangle

In acute △DEF, $d = 4.9$ cm, $f = 6.2$ cm, and ∠E = 64°. Solve △DEF. Round measures to the nearest degree or tenth of a centimetre, if necessary.

Solution

Draw a diagram and label the given information.



Since two sides and the contained angle are given, use the cosine law first to find side e.

Since I do not know the length of the side opposite the given angle, ∠E, I cannot use the sine law.

$e^2 = d^2 + f^2 - 2df(\cos E)$

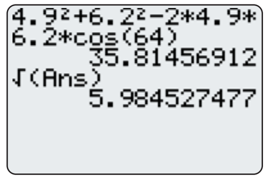
$e^2 = 4.9^2 + 6.2^2 - 2(4.9)(6.2)(\cos 64°)$

$e^2 = 35.814\ldots$

$e = \sqrt{35.814\ldots}$

$e \doteq 6.0$

Side e is about 6.0 cm.



Use the sine law to find the measure of one of the other angles.

$\frac{\sin D}{d} = \frac{\sin E}{e}$

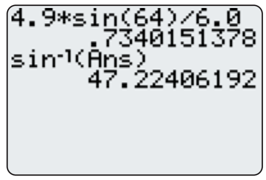
$\frac{\sin D}{4.9} = \frac{\sin 64°}{6.0}$

$\sin D = 4.9\left(\frac{\sin 64°}{6.0}\right)$

$\sin D = 0.7340\ldots$

$\angle D = \sin^{-1}(0.7340\ldots)$

$\angle D \doteq 47°$

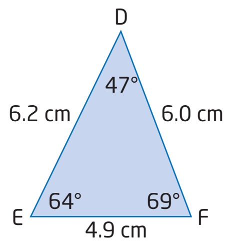


Determine the measure of the third angle.

$\angle F = 180° - 47° - 64°$

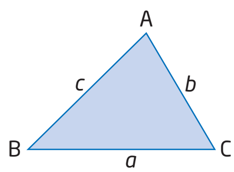
$= 69°$

△DEF has been solved for all side lengths and angle measures.



Key Concepts

- In an acute △ABC, the cosine law states that
  $a^2 = b^2 + c^2 - 2bc(\cos A)$
  $b^2 = c^2 + a^2 - 2ca(\cos B)$
  $c^2 = a^2 + b^2 - 2ab(\cos C)$



- The cosine law can be used to find a missing side if the other two sides and their contained angle are known.

Communicate Your Understanding

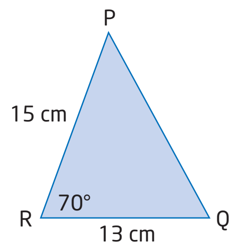
C1 **a)** What is meant by a contained angle in a triangle? Draw a diagram to illustrate your answer.

b) Why is this concept important when applying the cosine law?

C2 **a)** How are the cosine law and the Pythagorean theorem similar? How are they different?

b) Under what conditions can you apply
- the Pythagorean theorem?
- the cosine law?

C3 Describe the steps you would use to solve △PQR.

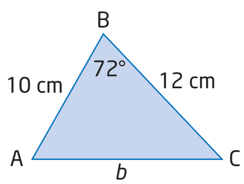


Practise

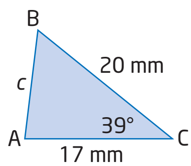
For help with questions 1 to 3, see Example 1.

1. Find the missing side length in each triangle, to the nearest unit.

a)

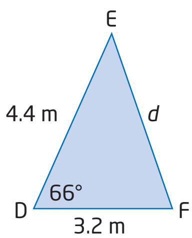


b)



2. Find the missing side length in each triangle, to the nearest tenth of a unit.

a)



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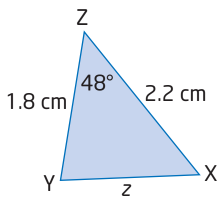

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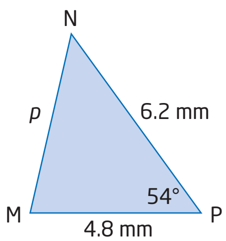


3. Sketch each triangle and use the given information to find the missing side length, to the nearest tenth of a unit.

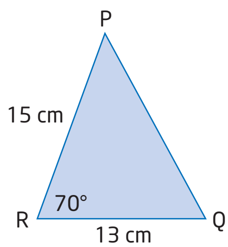
a) In acute △TUV, $t = 1.8$ cm, $v = 1.4$ cm, and $\angle U = 52°$.

b) In acute △DEF, $e = 1.1$ km, $f = 1.6$ km, and $\angle D = 74°$.

For help with questions 4 and 5, see Example 2.

4. Solve each triangle. Round answers to the nearest unit, if necessary.

a)



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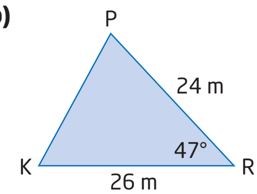
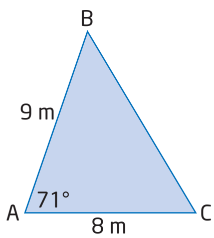


c)



5. Draw a diagram and label the given information. Then, solve each triangle. Round answers to the nearest unit, if necessary.

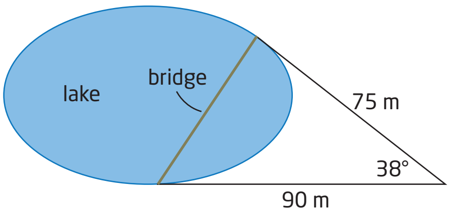
a) In acute △EFG, e = 5 cm, f = 6 cm, and ∠G = 63°.

b) In acute △WXY, w = 10 m, y = 11 m, and ∠X = 80°.

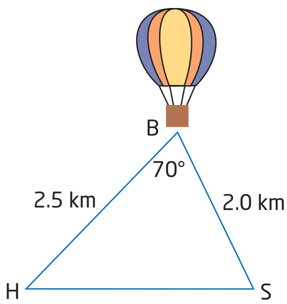
6. **Use Technology** Check your answers to question 5 using dynamic geometry software.

Connect and Apply

7. Find the length of the bridge, to the nearest metre.



8. Chandra is riding in a hot-air balloon and spots her house and her school. She estimates how far away they are from her, and the angle separating their lines of sight, as shown.

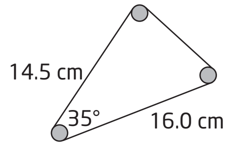


Use Chandra's estimated measures.

a) How far apart are Chandra's home and school, to the nearest tenth of a kilometre?

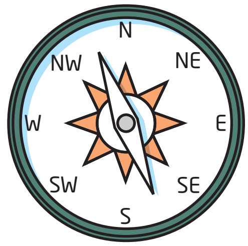
b) Chandra's mom is watching her from home, and her friends are watching from school. At what angle of elevation does Chandra appear to each of them, to the nearest degree?

9. A drive belt wraps around three pulleys, as shown.



Find the total length of the drive belt, to the nearest tenth of a centimetre. Ignore the curved sections.

10. Two ships leave port at the same time, travelling at the same speed of 10 knots. *Wavedancer* sails north, while *Ocean Princess* travels northeast.



a) How far apart are the two ships after 1 h?

b) How far apart are they after 2 h?

c) How do these answers change if *Wavedancer* travels twice as fast? Justify your answer.

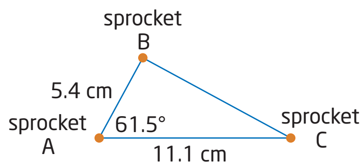
Did You Know?

A "knot" is short for a "nautical mile per hour."

1 knot = 1.852 km/h

1 nautical mile is the distance between two points on the equator that are separated by $\frac{1}{60}$ of a degree, making the circumference of Earth 21 600 nautical miles.

11. Connor is building a toy model of a track-type bulldozer. Three sprockets for one of the tracks are to be assembled as shown.



a) How far should sprocket C be placed from sprocket B, to the nearest tenth of a centimetre?

b) Find the interior angles of the triangle formed by these sprockets, to the nearest tenth of a degree.

12. **a)** Create a problem involving the cosine law for which the answer is 24 cm.

b) Solve the problem.

c) Trade with a partner and solve each other's problem. Check your solutions.

13. The airport at Goderich, Ontario, has two runways with lengths 1525 m and 915 m. The beginnings of the runways meet at an angle of 37°. The other ends of the runways are called the thresholds.

a) Draw a diagram and label the given information.

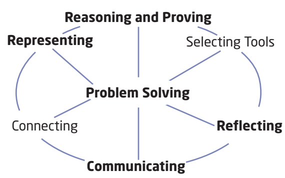
b) How far apart are the thresholds of the runways, to the nearest metre?

14. **Chapter Problem** While cruising at a steady speed of 400 km/h, you identify a storm cloud straight ahead 45 km away. To avoid turbulence, you start climbing at an angle of elevation of 15°. If you maintain this speed and direction for 6 min, how far will you be from the storm cloud? Round to the nearest kilometre.

15. Show that the cosine law simplifies to the Pythagorean theorem when the contained angle between the two known sides is 90°.

Achievement Check

16. One of the tallest totem poles in the world is located in Alert Bay, British Columbia. When the angle of elevation of the Sun is 62°, the totem pole casts a shadow of 30 m.



a) Suppose the totem stood vertical—how tall would it be, to the nearest tenth of a metre?

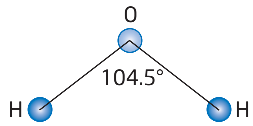
b) Suppose it was not quite vertical so that it made an angle of 89° with the ground. In this case, would your answer for its height be taller or shorter than your answer in part a)? Justify without calculations.

c) Calculate the difference between these two heights.

Extend

17. Design and carry out an investigation to determine if the cosine law holds true for obtuse triangles. Write a brief report of your findings.

18. Lee is building a scale model of a water molecule for his science project. The molecule consists of one oxygen atom and two hydrogen atoms, chemically bonded as shown.



Lee models the bond for each hydrogen atom with the oxygen atom using a 10-cm straw.

a) How far will the two hydrogen atoms be from each other, to the nearest tenth of a centimetre?

b) What angles will a line joining the two hydrogen atoms make with the lines of their chemical bonds?

19. Refer to question 9. Suppose that the measured distances are taken between the centres of each pulley, and that the diameter of each pulley is 2.8 cm. Find a more accurate total length of the drive belt. State any assumptions you make.

20. **Math Contest** Sam and Nick are both members of the basketball team, along with 10 other players. When the team is seated on the bench, what is the probability that Sam and Nick do not sit beside each other?

A $\frac{11}{12}$ **B** $\frac{1}{6}$

C $\frac{3}{4}$ **D** $\frac{5}{6}$

E $\frac{10}{11}$

21. **Math Contest** Show that for any ∠A, $(\sin A)^2 + (\cos A)^2 = 1$.