

## 8.3

## Find Angles Using the Cosine Law



You have learned when the sine law and cosine law can each be used to solve acute triangles.

What if you are given three side lengths in an acute triangle?

Runners in a 10-K race follow a triangular course. The three legs of the race, in order, are 3.8 km, 2.4 km, and 3.8 km. Can you apply the sine law or the cosine law to find the angle between the starting leg and the finishing leg?

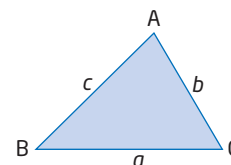
**Tools**

- protractor
- ruler

### Investigate

**How can you solve an acute triangle if three side lengths are given?**

1. **a)** Draw an acute scalene  $\triangle ABC$ .  
**b)** Measure the side lengths  $a$ ,  $b$ , and  $c$ .
2. Can you use the lengths of the three sides with the sine law to find the measure of one of the angles in  $\triangle ABC$ ? Explain.
3. **a)** Can you use the lengths of the three sides with the cosine law to find the measure of one of the angles in  $\triangle ABC$ ? Explain.  
**b)** Does the choice of angle matter? Explain.
4. **a)** Solve for one of the unknown angles using a trigonometric law and algebraic manipulation.  
**b)** Check your answer with a protractor.
5. **Reflect** Explain which trigonometric law can be used to find an angle measure if three side lengths are known in an acute triangle. How would you solve the equation?

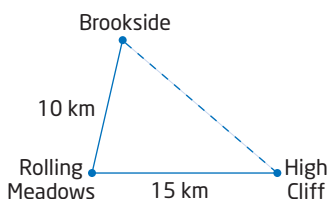


If you know any three measurements of an acute triangle, including at least one side length, you can find the remaining measurements using the sine law and the cosine law. The choice depends on which law allows you to relate the three given measurements to one unknown measurement.

### Example 1 Find an Angle Using the Cosine Law

Three towns are connected by two roads, as shown.

A third road is planned that will directly connect Brookside and High Cliff, which are 16.5 km apart. Find the angle, to the nearest tenth of a degree, between the new road and the existing road from

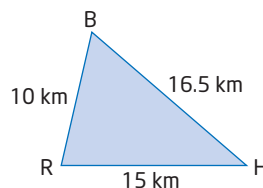


- Rolling Meadows to Brookside
- Rolling Meadows to High Cliff

#### Solution

Draw a simplified diagram and label it with the given information.

The new road will make unknown angles at B and H.



- Use the cosine law to find  $\angle B$ .

#### Method 1: Substitute, Then Rearrange

$$b^2 = h^2 + r^2 - 2hr(\cos B)$$

$$15^2 = 10^2 + 16.5^2 - 2(10)(16.5)(\cos B)$$

$$225 = 100 + 272.25 - 330(\cos B) \quad \text{Simplify.}$$

$$225 = 372.25 - 330(\cos B)$$

$$225 - 372.25 = -330(\cos B) \quad \text{Isolate the term containing } \cos B.$$

$$-147.25 = -330(\cos B)$$

$$\frac{-147.25}{-330} = \frac{-330(\cos B)}{-330} \quad \text{Divide both sides by } -330.$$

$$\frac{147.25}{330} = \cos B$$

$$\cos^{-1}\left(\frac{147.25}{330}\right) = \angle B$$

Apply the inverse cosine to find  $\angle B$ .

$$63.5^\circ \doteq \angle B$$

The new road will make an angle of approximately  $63.5^\circ$  with the existing road from Rolling Meadows to Brookside.

### Method 2: Rearrange, Then Substitute

$$b^2 = h^2 + r^2 - 2hr(\cos B)$$

$$b^2 - h^2 - r^2 = -2hr(\cos B) \quad \text{Isolate the term containing } \cos B.$$

$$\frac{b^2 - h^2 - r^2}{-2hr} = \frac{-2hr(\cos B)}{-2hr} \quad \text{Divide both sides by } -2hr.$$

$$\frac{b^2 - h^2 - r^2}{-2hr} = \cos B$$

$$\frac{15^2 - 10^2 - 16.5^2}{-2(10)(16.5)} = \cos B \quad \text{Substitute the known information.}$$

$$\frac{-147.25}{-330} = \cos B \quad \text{Simplify.}$$

$$\cos^{-1}\left(\frac{147.25}{330}\right) = \angle B \quad \text{Apply the inverse cosine to find } \angle B.$$

$$63.5 \doteq \angle B$$

The new road will make an angle of approximately  $63.5^\circ$  with the existing road from Rolling Meadows to Brookside.

### Method 3: Rearrange, Then Substitute, Using a Computer Algebra System

Ensure that angle measurements are calculated in degrees.

Let  $\angle B$  be represented by  $x$ .

In the Home screen, type the cosine law:

$$b^2 = h^2 + r^2 - 2hr(\cos x)$$

TI-89 calculator screen showing the cosine law equation entered in the Home screen. The equation is  $b^2 = h^2 + r^2 - 2 \cdot h \cdot r \cdot \cos(x)$ . The screen also shows the result  $b^2 = r^2 - 2 \cdot h \cdot r \cdot \cos(x) + h^2$  and the input  $b^2 = h^2 + r^2 - 2 \cdot h \cdot r \cdot \cos(x)$ . The status bar at the bottom shows MAIN, DEGRAUTO, FUNC, and 2/30.

Subtract  $h^2$  and  $r^2$  to isolate the term with  $\cos x$ .

TI-89 calculator screen showing the cosine law equation with terms rearranged. The equation is  $b^2 = h^2 + r^2 - 2 \cdot h \cdot r \cdot \cos(x)$ . The screen also shows the result  $b^2 = r^2 - 2 \cdot h \cdot r \cdot \cos(x) + h^2$  and the input  $(b^2 = h^2 + r^2 - 2 \cdot h \cdot r \cdot \cos(x)) - r^2 + b^2 - h^2 = -2 \cdot h \cdot r \cdot \cos(x)$ . The status bar at the bottom shows MAIN, DEGRAUTO, FUNC, and 3/30.

Divide by the expression  $-2hr$ .

TI-89 calculator screen showing the cosine law equation divided by  $-2hr$ . The equation is  $-r^2 + b^2 - h^2 = -2 \cdot h \cdot r \cdot \cos(x)$ . The screen also shows the result  $-r^2 + b^2 - h^2 = -2 \cdot h \cdot r \cdot \cos(x)$  and the input  $\frac{-r^2 + b^2 - h^2}{-2 \cdot h \cdot r} = \cos(x)$ . The status bar at the bottom shows MAIN, DEGRAUTO, FUNC, and 4/30.

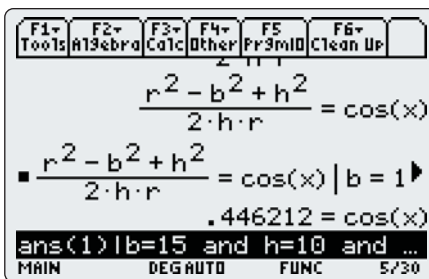
### Technology Tip

To set the angle mode to degrees on a TI-89 calculator:

- Press **MODE**. Cursor down to the **Angle** mode setting.
- Press **▶**, and select **2:DEGREE**.
- Press **ENTER** to save the settings.

Substitute the values for  $b$ ,  $h$ , and  $r$ .

- Press  $\text{2nd}$   $(-)$  for [ANS], followed by the *such that* symbol,  $(|)$ .
- Type  $b = 15$ . Press  $\text{2nd}$  [MATH] to display the MATH menu, select **8:Test**, and then **8:and**.
- Type  $h = 10$ . Press  $\text{2nd}$  [MATH], select **8:Test**, and then **8:and**.
- Type  $r = 16.5$  and press  $\text{ENTER}$ .



Find the angle.

- Press  $\text{2nd}$   $(\cos^{-1})$  [COS<sup>-1</sup>], and then  $\text{2nd}$  [ANS], followed by  $(\text{ )}$ .
- Press  $\text{ENTER}$ .



The new road will make an angle of approximately  $63.5^\circ$  with the existing road from Rolling Meadows to Brookside.

- b)** You could apply the cosine law a second time to find  $\angle H$ . However, now you have enough information to apply the sine law. The sine law requires fewer calculation steps.

$$\frac{\sin H}{h} = \frac{\sin B}{b}$$

$$\frac{\sin H}{10} = \frac{\sin 63.5^\circ}{15}$$

$$\sin H = 10 \left( \frac{\sin 63.5^\circ}{15} \right)$$

$$\sin H = 0.5966\dots$$

$$\angle H = \sin^{-1}(0.5966\dots)$$

$$\angle H \doteq 36.6^\circ$$

The new road will make an angle of approximately  $36.6^\circ$  with the existing road from Rolling Meadows to High Cliff.

## Example 2 Solve an Acute Triangle

Runners in a 10-K race follow a triangular course. The three legs of the race, in order, are 3.8 km, 2.4 km, and 3.8 km. Find the angle between the legs in each pair, to the nearest degree.

- first and second
- second and third
- first and third

## Solution

Draw a diagram and label it with the known information.

- a) Since the measures of three sides are given, use the cosine law to find  $\angle B$ .

$$b^2 = a^2 + c^2 - 2ac(\cos B)$$

$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$

Rearrange to isolate  $\cos B$ .

$$\cos B = \frac{3.8^2 - 2.4^2 - 3.8^2}{-2(2.4)(3.8)}$$

Substitute known values.

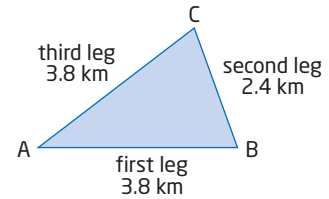
$$\cos B = \frac{-5.76}{-18.24}$$

Simplify.

$$\angle B = \cos^{-1}\left(\frac{5.76}{18.24}\right)$$

Solve for  $\angle B$ .

$$\angle B \doteq 72^\circ$$



The angle between the first and second legs of the race is  $72^\circ$ .

- b) **Method 1: Apply the Sine Law**

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin C}{3.8} = \frac{\sin 72^\circ}{3.8}$$

$$\sin C = 3.8 \left( \frac{\sin 72^\circ}{3.8} \right)$$

$$\sin C = \sin 72^\circ$$

Therefore,  $\angle C = 72^\circ$ .

The angle between the second and third legs of the race is  $72^\circ$ .

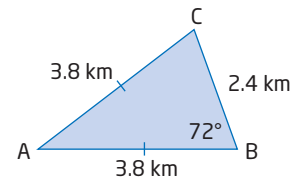
### Method 2: Apply Geometric Reasoning

Since  $AC = AB$ ,  $\triangle ABC$  is isosceles.

The two angles opposite the two equal sides of an isosceles triangle are equal.

Since  $c = b$ , then  $\angle C = \angle B$ . Therefore,  $\angle C = 72^\circ$ .

The angle between the second and third legs of the race is  $72^\circ$ .



- c) Use the two known angles to find  $\angle A$ .

$$\begin{aligned}\angle A &= 180^\circ - 72^\circ - 72^\circ \\ &= 36^\circ\end{aligned}$$

The angle between the first and third legs of the race is  $36^\circ$ .

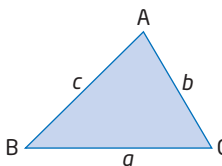
## Key Concepts

- You can rearrange the cosine law to find an angle if you know three side lengths of an acute  $\triangle ABC$ .

For example, to find the measure of  $\angle B$ , rearrange the appropriate form of the cosine law.

$$b^2 = a^2 + c^2 - 2ac(\cos B) \Rightarrow \cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$

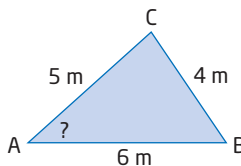
- Once you have found one angle, you can apply the sine law to find a second angle.
- When applying trigonometry to solve problems involving acute triangles, there is often more than one valid strategy.



## Communicate Your Understanding

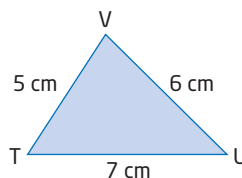
- C1** When can you use the sine law to find an unknown angle in an acute triangle? When must you use the cosine law? Draw diagrams to support your answer.

- C2** The following steps show how the cosine law can be rearranged to find  $\angle A$ . Copy the steps and write a short explanation beside each one.



Steps	Explanation
$a^2 = b^2 + c^2 - 2bc(\cos A)$	Write the cosine law that includes $\angle A$ .
$a^2 - b^2 - c^2 = -2bc(\cos A)$	
$\frac{a^2 - b^2 - c^2}{-2bc} = \frac{-2bc(\cos A)}{-2bc}$	
$\frac{a^2 - b^2 - c^2}{-2bc} = \cos A$	
$\frac{4^2 - 5^2 - 6^2}{-2(5)(6)} = \cos A$	
$\frac{-45}{-60} = \cos A$	
$\cos^{-1}\left(\frac{45}{60}\right) = \angle A$	
$41^\circ \doteq \angle A$	

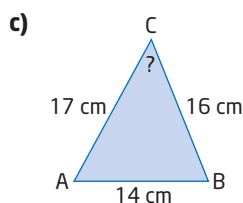
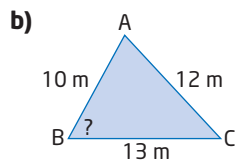
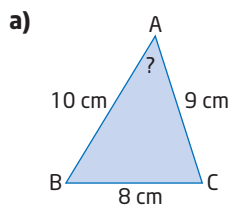
- C3** a) Describe the steps you would take to solve  $\triangle TUV$ .  
 b) Describe a different set of steps that will also work.



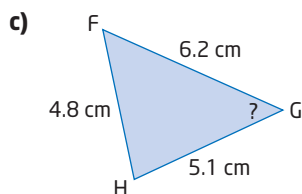
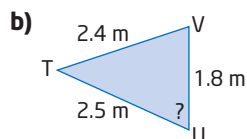
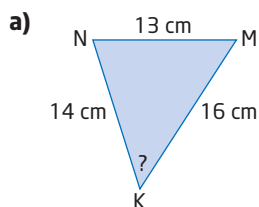
## Practise

For help with questions 1 to 3, see Example 1.

1. Solve for the indicated angle, to the nearest degree.



2. Solve for the indicated angle, to the nearest degree.



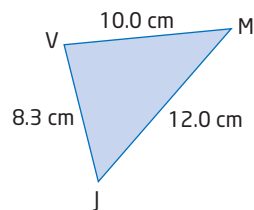
3. Sketch each triangle. Then, use the given information to find the indicated angle, to the nearest degree.

- a) In acute  $\triangle ARD$ ,  $a = 170$  mm,  $r = 190$  mm, and  $d = 210$  mm. Find  $\angle D$ .

- b) In acute  $\triangle HWN$ ,  $h = 1.4$  km,  $w = 1.7$  km, and  $n = 1.2$  km. Find  $\angle W$ .

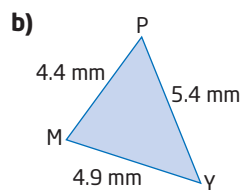
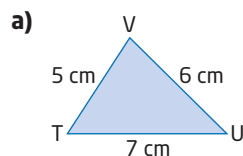
For help with questions 4 to 6, see Example 2.

4. Consider  $\triangle JVM$ .



- a) Follow these steps in order to solve  $\triangle JVM$ . Round answers to the nearest tenth of a degree.
- Use the cosine law to find  $\angle J$ .
  - Use the cosine law to find  $\angle V$ .
  - Use the cosine law to find  $\angle M$ .
- b) Solve  $\triangle JVM$  using a more efficient method.
- c) Compare your answers in parts a) and b). Explain why your method is more efficient.

5. Solve each triangle. Round answers to the nearest tenth of a degree.



6. Sketch each triangle and label it with the given information. Then, solve the triangle. Round answers to the nearest tenth of a degree.

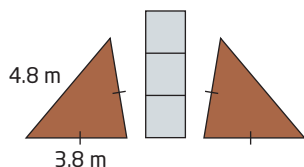
- a) In acute  $\triangle NBG$ ,  $n = 15$  m,  $b = 14$  m, and  $g = 12$  m.

- b) In acute  $\triangle DRT$ ,  $d = 5.0$  km,  $r = 3.8$  km, and  $t = 4.6$  km.

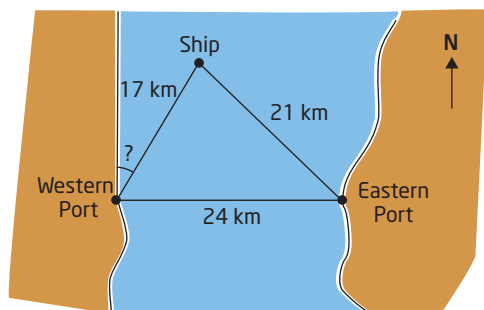
7. **Use Technology** Check your answers to question 6 using dynamic geometry software.

## Connect and Apply

8. Ling is designing a garden for her backyard. It will consist of two congruent triangular flower beds on either side of a path, as shown.

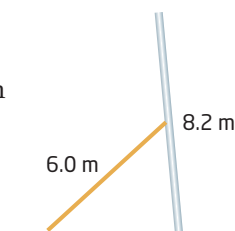


- Find the interior angles of the flower beds, to the nearest degree.
  - Find the total area of the flower beds, to the nearest square metre.
9. A distress signal is received from a ship that is 21 km from one port and 17 km from another port. The eastern port is 24 km directly east of the western port. At what angle to the western shoreline should the ship head, in order to dock at the western port? Round to the nearest degree.



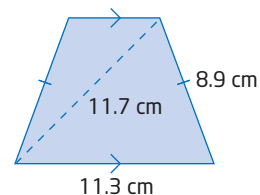
10. A leaning pole is braced at its midpoint, as shown. The pole is 8.2 m long, and the bracing beam is 6.0 m long.

The foot of the beam is placed 5.0 m from the base of the pole.



- Determine, to the nearest degree,
- the angle the pole makes with the ground
  - the angle the beam makes with the ground
  - the angle the beam makes with the pole

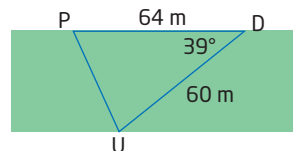
11. Find the interior angles of the isosceles trapezoid, to the nearest tenth of a degree.



12. Create a problem involving the cosine law for which the answer is  $72^\circ$ .
13. **Chapter Problem** You and two of your team are to fly in a V-formation such that the distances between you are 80 m, 80 m, and 100 m. Find the angles that illustrate how the three aircraft should be arranged, to the nearest tenth of a degree. Is there more than one possible solution? Explain.

## Achievement Check

14. Pinder, Dino, and Ursala all live on the edge of a park, as shown.



They agree to meet at one of their homes to study for a trigonometry test.

- Is it possible to solve this triangle? Justify your response.
- What tools will you use?
- Whose home should they pick in order to minimize travel time? State any assumptions you make.

## Extend

15. a) Acute  $\triangle ABC$  is isosceles, with  $b = c$ . Show that  $\cos A = 1 - \frac{a^2}{2b^2}$ .
- b) Solve  $\triangle ABC$ , if the equal sides are 1.5 cm and the third side is 0.8 cm.
16. Draw an equilateral triangle and label its sides  $a$ . Mark one of the angles  $60^\circ$ . Use the cosine law to prove that  $\cos 60^\circ = \frac{1}{2}$ .