4.2 Quadratic Relations

The Galleria, in BCE Place in Toronto, has many arches. These curved structures are used to span a space while supporting weight. Just as a linear relation can be modelled with a linear equation, some non-linear relations, such as the shape of an arch, can be modelled using non-linear equations.

Tools

- grid paper

Investigate A

How can you compare relations of the form $y = ax^2 + bx + c$?

1. Make a table of values for each relation, using integer values of x from -3 to $+3$.
   a) $y = x^2$ **b)** $y = 2x^2$ **c)** $y = x^2 + 2x + 3$
   d) $y = -x^2$ **e)** $y = -0.5x^2 + 3$
2. Graph all the relations in step 1 on the same set of axes. Plot each set of ordered pairs and draw a smooth curve through the points.
3. **Reflect** Describe the graphs you created in as many ways as you can. What is similar about the graphs? What is different?

quadratic relation

- a relation whose equation is in the form $y = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$

parabola

- the graph of a quadratic relation, which is U-shaped and symmetrical

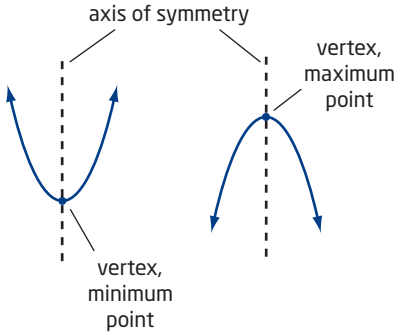
vertex

- the point on a parabola where the curve changes direction
- the maximum point if the parabola opens down
- the minimum point if the parabola opens up

axis of symmetry

- the line that divides a figure into two congruent parts

The relation described by $y = ax^2 + bx + c$ is called a **quadratic relation**. The graph of a quadratic relation is called a **parabola**. A parabola has a minimum point or a maximum point called the **vertex**. It is also symmetrical about a vertical line drawn through the vertex, called the **axis of symmetry**.



Investigate B

How can you use finite differences to determine if a relation is linear or quadratic?

finite differences

- differences found from the y-values in tables with evenly spaced x-values
- first differences are the differences between consecutive y-values, second differences are the differences between consecutive first differences, and so on

1. Copy and complete the table for each linear relation. Calculate the y-values. Then, calculate the first differences by subtracting consecutive y-values.

a) $y = 2x - 5$

x	y	First Differences
−2	−9	
−1	−7	−7 − (−9) = 2
0		
1		
2		

b) $y = -6x + 2$

x	y	First Differences
−2	14	
−1	8	8 − 14 = −6
0		
1		
2		

2. What is true about the first differences for a linear relation?

3. Copy and complete the table for each quadratic relation. Calculate the y-values and first differences. Then, calculate the second differences by subtracting successive first differences.

a) $y = x^2 - 4$

x	y	First Differences	Second Differences
−2	0		
−1	−3	−3 − 0 = −3	
0	−4	−4 − (−3) = −1	−1 − (−3) = 2
1			
2			

b) $y = 2x^2 + 3x - 1$

x	y	First Differences	Second Differences
−2	1		
−1	−2		
0			
1			
2			

4. a) What is true about the first differences for a quadratic relation?

b) What is true about the second differences for a quadratic relation?

5. Reflect Write a rule for using finite differences to determine whether a relation is linear or quadratic.

Example Galleria Arches

Each arch in the BCE Place Galleria can be approximated by the relation $y = -0.55x^2 + 26$, where y is the height, in metres, above the floor and x is the width, in metres, from the centre of the hallway.

a) Graph the quadratic relation.

b) Describe the shape of the arch.

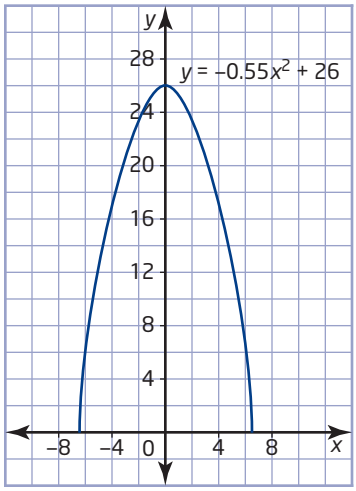
c) How tall and wide is the arch?

Solution

a) Method 1: Use Pencil and Paper

Use a table of values to help you sketch the graph.

x	y
−6	6.2
−4	17.2
−2	23.8
0	26.0
2	23.8
4	17.2
6	6.2

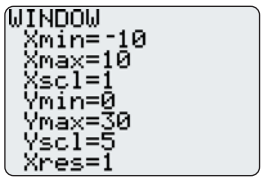


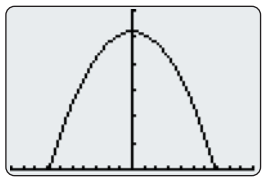
Method 2: Use a Graphing Calculator

Enter the equation using [Y=].

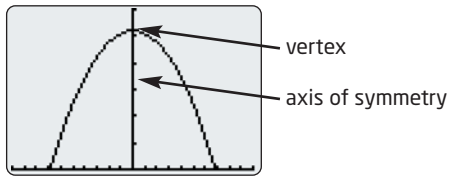
Press [WINDOW] and enter the settings shown.

Then, press [GRAPH].

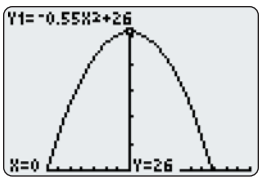




b) You can see that the shape of the arch is parabolic. The parabola is symmetrical about a vertical line, the y-axis. The graph has a maximum point.



c) You can read the maximum y-value from the pencil-and-paper graph or use the TRACE feature on the graphing calculator.



Since the maximum value of y is 26, the height of each arch is 26 m.

The x-axis represents the floor of the hallway. The width of each arch is the difference between the two x-intercepts. From the pencil-and-paper graph, the x-intercepts appear to be about 7 and -7. Use the TRACE feature on the graphing calculator to find that the curve crosses the x-axis at about -6.9 and $+6.9$.

$6.9 - (-6.9) = 13.8$

The width of each arch is about 13.8 m.

I can see that the maximum occurs when $x = 0$. From the equation, when $x = 0$, $y = 26$.

Technology Tip

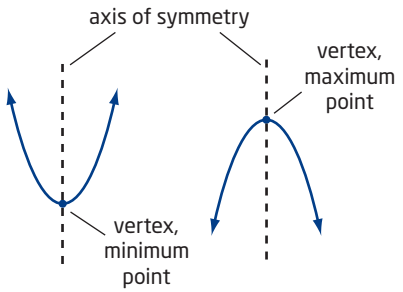
You can get a better approximation of the x-intercepts by zooming in.

- Position the cursor near one of the x-intercepts.
- Press ZOOM, select **2:Zoom In**, and then press ENTER.
- Press TRACE and reposition the cursor.

The accuracy of the approximation improves each time you repeat these steps.

Key Concepts

- The relation defined by $y = ax^2 + bx + c$ is a quadratic relation.
- The graph of a quadratic relation is called a parabola.
- The vertex of a parabola is either the minimum point or the maximum point on the graph.



- A parabola is symmetric about a vertical line that passes through the vertex. This line is the axis of symmetry.
- If a relation is quadratic, the second differences are constant, but the first differences are not.

Communicate Your Understanding

C1 El-Noor used the following incorrect technique to determine that the relation is not quadratic. Explain the flaw in his reasoning.

x	y	First Differences	Second Differences
−3	13		
		−10	
−2	3		2
		−8	
0	−5		10
		2	
1	−3		4
		6	
2	3		18
		24	
4	27		

C2 In Section 4.1, Investigate Part A, you found that the relationship between thumb length and palm area is non-linear. Is the relation quadratic? Explain.

Practise

For help with questions 1 and 2, see the Example.

1. The table shows the path of a ball, where x is the horizontal distance, in metres, and h is the height, in metres, above the ground.

x	h
0	1
1	8
2	13
3	16
4	17
5	16
6	13
7	8
8	1

a) Sketch a graph of the quadratic relation.

b) Describe the flight path of the ball. Identify the axis of symmetry and the vertex.

c) What is the maximum height that the ball reached?

d) Verify that $h = -x^2 + 8x + 1$ can be used to model the flight path of the ball.

2. The underside of a bridge is an arch that can be approximated by the relation $y = -0.1x^2 + 10$, where y is the height, in metres, above the ground and x is the width, in metres, from the centre of the bridge.

a) Graph the quadratic relation.

b) Describe the shape of the arch.

c) How tall and wide is the arch?

3. Use finite differences to determine whether each relation is linear, quadratic, or neither.

a)

x	y
0	4
1	5
2	6
3	7
4	8

b)

x	y
0	3
1	4
2	7
3	12
4	19

c)

x	y
1	0
3	1
5	8
7	27
9	64

d)

x	y
–2	6
1	0
4	12
7	42
10	90

Connect and Apply

4. This section has photographs of parabolic arches in architecture, furniture, bridge design, and nature. Find five more examples of parabolic arches. Some possible sources are the Internet, personal surroundings, or print-based material. Explain how you determined that your examples are parabolic.

5. The parabolic shape of the Humber River Pedestrian Bridge in Toronto can be approximated by the equation $h = -\frac{1}{144}x^2 + \frac{5}{6}x$, where x is the horizontal distance, in metres, from one end and h is the height, in metres, above the water.

a) Graph the quadratic relation with or without technology.

b) What is the height of the bridge 12 m horizontally from one end?

c) How wide is the bridge at its base?

d) What is the maximum height of the bridge? At what horizontal distance does it reach that height?

e) Identify the axis of symmetry of the bridge.

6. Use Technology A ball is thrown upward at an initial velocity of 15 m/s, from a height of 1.5 m. The height, h, in metres, of the ball above the ground after t seconds can be found using the relation $h = -4.9t^2 + 15t + 1.5$.

a) Graph this relation using a graphing calculator.

b) Describe the relationship between time and height.

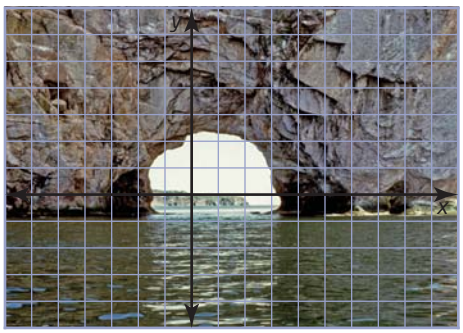
c) Repeat parts a) and b) for a ball thrown upward on the Moon, with height defined by the relation $h = -0.81t^2 + 15t + 1.5$.

d) Repeat parts a) and b) for a ball thrown upward on Jupiter, with height defined by the relation $h = -11.55t^2 + 15t + 1.5$.

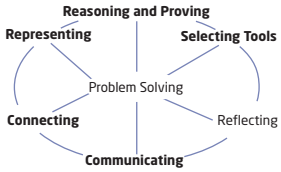
e) Compare the results from the three locations.

7. Chapter Problem A city opened a new landfill site in 2000. In Section 4.1, question 6, you created a table of values showing the total mass of garbage in the landfill for each year from 2000 to 2007. Use your table and finite differences to determine if the relationship more closely models a linear or a quadratic relation. Justify your decision.

8. Percé Rock is located at the eastern tip of Quebec's Gaspé Peninsula. Make a table of values of at least seven points, so that the x-values are equally spaced. Use finite differences to determine how close the arch is to a parabola.



Achievement Check



9. The path of a rocket fired at a Canada Day fireworks display is given by $h = -4.9t^2 + 19.6t + 0.4$, where h is the height, in metres, of the rocket above the ground and t is the time, in seconds.

a) Make a table of values for $t = 0$ to $t = 4$.

b) Make a table of first and second differences. What conclusion can you make?

c) Draw a graph of the path of the rocket using the table of values from part a) or graphing technology. Describe the path of the rocket.

d) How high above the ground was the rocket when it was set off? Explain your answer.

Extend

10. The flow rate of water through a garden hose depends on the water pressure and the diameter of the hose opening. At a normal water pressure of 345 kPa, the flow rate can be calculated using the formula $r = 2d^2$, where d is the diameter, in centimetres, of the hose opening and r is the flow rate, in litres per second. How long would it take to fill a 200-L barrel using a hose with a 0.3-cm-diameter opening?

11. The sum of the first n natural numbers is a quadratic relation. Determine that relation and verify it for the first six natural numbers.