4.3 Investigate Transformations of Quadratics

When police investigate car accidents, they often measure the length of a tire skid mark to determine the speed of the car before braking. Such calculations involve quadratic relations. On dry pavement, the length of a skid mark, l, is related to the speed of the car, s, before braking by the relation $l = 0.04s^2$.

Investigate

How do transformations of the graph of $y = x^2$ affect the equation?

Tools

- TI-83 Plus or TI-84 Plus graphing calculator
- grid paper

A: Compare the Graphs of $y = x^2$ and $y = x^2 + k$

1. First, clear any graphed equations.

- Press [Y=] and use the [CLEAR] key to remove any equations.
- Make sure **Plot1**, **Plot2**, and **Plot3** are not highlighted. If they are, use the [▶], [◀], [▲], and [▼] keys to move to each and press [ENTER].

2. Use a standard window.

- Press [ZOOM] and select **6:ZStandard.**
- You can view the window settings by pressing [WINDOW].

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WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```

3. Graph the equation $y = x^2$ as **Y1**.

- Press [Y=]. Beside **Y1=**, press [X,T,θ,n] [x^2].
- Press [GRAPH] to view the parabola.

4. Enter $y = x^2 + 2$ as **Y2** and $y = x^2 - 4$ as **Y3**. Press [GRAPH].

5. **a)** Sketch all three graphs on the same set of axes. Label each parabola with its equation.
   b) Describe the transformations.
   c) Without using a graphing calculator, sketch the graph of $y = x^2 - 8$.

6. **Reflect** Describe how the value of k in $y = x^2 + k$ changes the graph of $y = x^2$.

B: Compare the Graphs of $y = x^2$ and $y = ax^2$

1. **a)** Clear the equations from **Y2=** and **Y3=**, but keep $y = x^2$ in **Y1**.
   b) Graph the equations $y = 2x^2$ and $y = 3x^2$.

2. **a)** Sketch all three graphs on the same set of axes. Label each parabola with its equation.
   b) Describe the transformations.

3. Without using a graphing calculator, sketch the graph of $y = 4x^2$.

4. Repeat steps 1 and 2 for the equations $y = \frac{1}{2}x^2$ and $y = \frac{1}{4}x^2$.

5. Without using a graphing calculator, sketch the graph of $y = \frac{1}{3}x^2$.

6. Repeat steps 1 and 2 for the equations $y = -2x^2$ and $y = -0.5x^2$.

7. Without using a graphing calculator, sketch the graph of $y = -3x^2$.

8. **Reflect** Describe how the value of a in $y = ax^2$ changes the graph of $y = x^2$.

C: Compare the Graphs of $y = x^2$ and $y = (x - h)^2$

1. **a)** Clear all equations except $y = x^2$.
   b) Graph the equations $y = (x - 2)^2$ and $y = (x - 5)^2$.

2. **a)** Sketch all three graphs on the same set of axes. Label each parabola with its equation.
   b) Describe the transformations.

3. Without using a graphing calculator, sketch the graph of $y = (x - 3)^2$.

4. Repeat steps 1 and 2 using the equations $y = (x + 2)^2$ and $y = (x + 5)^2$.

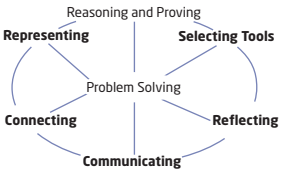
5. Without using a graphing calculator, sketch the graph of $y = (x + 3)^2$.

6. **Reflect** Describe how the value of h in $y = (x - h)^2$ changes the graph of $y = x^2$.

Technology Tip

A table of values can help you sketch the graph of an equation entered using Y=. You can specify how a table of values is set up.

- Press 2nd [TBLSET] to display the **TABLE SETUP** screen. Make sure both **Indpnt** and **Depend** are set to **Auto**. Enter the desired starting x-value (**TblStart**) and x increment (**ΔTbl**). For example, try **TblStart=−10** and **ΔTbl=1**.
- Press 2nd [TABLE] to view the table of values.



Example Falling Stone

A stone is dropped from the top of a 50-m cliff above a river. Its height, y, in metres, above the water can be estimated using the relation $y = -4.9x^2 + 50$, where x is the time, in seconds.

a) Graph the relation.

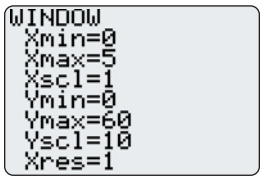
b) Find the intercepts. What do they represent?

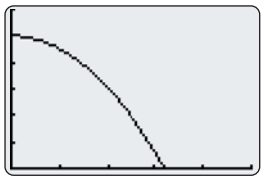
c) How would the equation change if the stone were dropped from a 75-m cliff instead of a 50-m cliff?

d) For what values of x is each equation valid?

Solution

a) Use a graphing calculator with the window settings shown.





Since height and time cannot be negative, this graph shows only part of a parabola.

b) For the y-intercept, let $x = 0$.

$y = -4.9(0)^2 + 50$

$= 50$

I can see that the y-intercept is 50 from the graph.

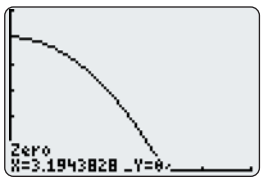
The y-intercept is 50. This represents the height from which the stone was dropped, 50 m above the water.

zero

- a value of x for which a relation has a value of 0
- corresponds to an x-intercept of the graph of the relation

To find the x-intercept, or **zero**, of the relation use the Zero operation on a graphing calculator.

- Press [2nd][CALC] to display the **CALCULATE** menu, and select **2:zero**.
- Move the cursor to the left of the x-intercept and press [ENTER].
- Move the cursor to the right of the x-intercept and press [ENTER].
- Press [ENTER] again.

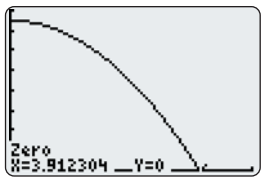


The x-intercept is approximately 3.19.

This represents the time when the stone hits the water, 3.19 s.

c) The constant term would be 75 instead of 50. The equation would change to $y = -4.9x^2 + 75$.

d) The original equation $y = -4.9x^2 + 50$ is valid for $0 \leq x \leq 3.19$ (approximately).

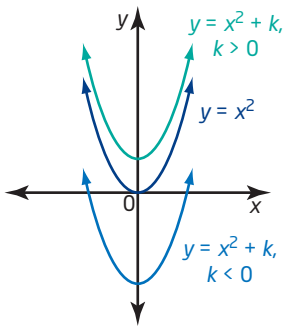


Graph the new equation and use the Zero operation. You will need to change the WINDOW settings to be able to see the whole graph.

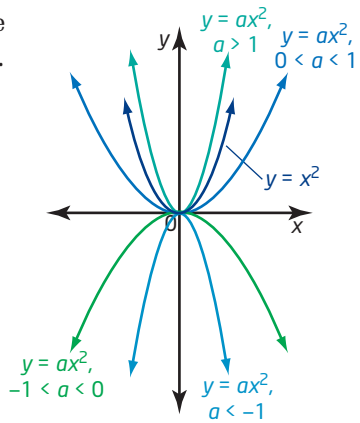
The new equation $y = -4.9x^2 + 75$ is valid for $0 \leq x \leq 3.91$ (approximately).

Key Concepts

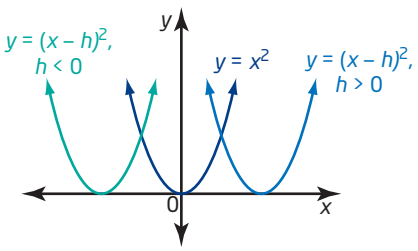
- To graph $y = x^2 + k$, translate the graph of $y = x^2$ vertically k units.
  - If $k > 0$, then the graph is translated k units upward.
  - If $k < 0$, then the graph is translated k units downward.



- To graph $y = ax^2$, stretch or compress the graph of $y = x^2$ vertically by a factor of a.
  - If $a < 0$, the parabola is reflected in the x-axis.
  - If $a > 1$ or $a < -1$, then the graph is stretched vertically (narrows).
  - If $-1 < a < 0$ or $0 < a < 1$, then the graph is compressed vertically (widens).



- To graph $y = (x - h)^2$, translate the graph of $y = x^2$ horizontally h units.
  - If $h > 0$, then the graph is translated h units to the right.
  - If $h < 0$, then the graph is translated h units to the left.



Communicate Your Understanding

C1 How do the graphs of $y = 2x^2$ and $y = -2x^2$ compare? Explain the similarities and the differences.

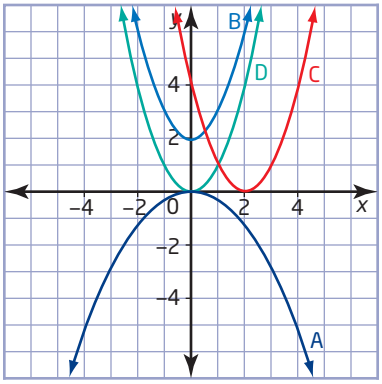
C2 Match each graph with the appropriate equation. Explain your reasoning.

a) $y = x^2 + 2$

b) $y = -\frac{1}{3}x^2$

c) $y = x^2$

d) $y = (x - 2)^2$



Practise

For help with questions 1 to 4, see the Investigate.

1. Sketch graphs of these three quadratic relations on the same set of axes.

a) $y = -3x^2$ **b)** $y = \frac{1}{4}x^2$

c) $y = -\frac{1}{4}x^2$

2. Sketch graphs of these three quadratic relations on the same set of axes.

a) $y = (x - 9)^2$ **b)** $y = (x + 2)^2$

c) $y = (x - 5)^2$

3. Sketch graphs of these three quadratic relations on the same set of axes.

a) $y = x^2 + 8$ **b)** $y = x^2 - 5$

c) $y = x^2 - 10$

4. Sketch the graph of each parabola. Label at least three points on the parabola. Describe the transformation from the graph of $y = x^2$.

a) $y = 4x^2$ **b)** $y = \frac{2}{3}x^2$

c) $y = x^2 - 5$ **d)** $y = (x - 8)^2$

e) $y = -\frac{1}{2}x^2$ **f)** $y = (x + 3)^2$

g) $y = x^2 + 0.5$ **h)** $y = -x^2 + 2$

5. a) Make tables of values for $y = x^2$, $y = 2x^2$, $y = x^2 + 1$, and $y = (x - 3)^2$.

b) Compare the y-values for $y = x^2$ and $y = 2x^2$.

c) Compare the y-values for $y = x^2$ and $y = x^2 + 1$.

d) Compare the y-values for $y = x^2$ and $y = (x - 3)^2$.

Connect and Apply

6. Write an equation for the quadratic relation that results from each transformation.

a) The graph of $y = x^2$ is translated 6 units upward.

b) The graph of $y = x^2$ is translated 4 units downward.

7. Write an equation for the quadratic relation that results from each transformation.

a) The graph of $y = x^2$ is translated 7 units to the left.

b) The graph of $y = x^2$ is translated 5 units to the right.

c) The graph of $y = x^2$ is translated 8 units to the left.

d) The graph of $y = x^2$ is translated 3 units to the right.

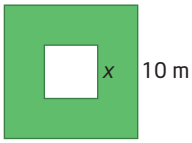
8. Write an equation for the quadratic relation that results from each transformation.

a) The graph of $y = x^2$ is stretched vertically by a factor of 8.

b) The graph of $y = x^2$ is compressed vertically by a factor of $\frac{1}{5}$.

For help with question 9, see the Example.

9. The grass in the backyard of a house is a square with side length 10 m. A square patio is placed in the centre. If the side length, in metres, of the patio is x, then the area of grass remaining is given by the relation $A = -x^2 + 100$.



a) Graph the relation.

b) Find the intercepts. What do they represent?

c) How does the equation change if the grass in the backyard of a house is a square with side length 12 m?

d) For what values of x is each equation valid?

10. The relation $l = 0.04s^2$ can be used to calculate the length, l, in metres, of the skid mark for a car travelling at a speed, s, in kilometres per hour, on dry pavement before braking.

a) What is the length of the skid mark for a car travelling at 50 km/h? 100 km/h?

b) How do the results in part a) compare?

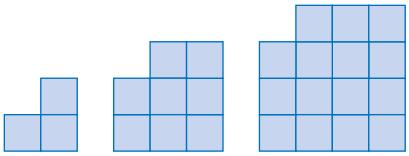
c) For what values of s is this model valid?

d) How would the skid marks and the equation change if the pavement were wet?

Did You Know?

A Technical Collision Investigator or Reconstructionist is a specially trained police officer who investigates serious traffic accidents. These officers collect and interpret evidence to determine the cause of the collision and if any charges should be laid.

11. The first three diagrams in a pattern are shown. Each square has a side length of 1 unit.



a) Make a table comparing base length and area. Use finite differences to determine whether the relation is linear, quadratic, or neither.

b) Determine an equation for the relationship between the base length and the area.

c) Describe the transformation from the graph of $y = x^2$.

Extend

12. The transformations to graph $y = ax^2$ and $y = x^2 + k$ both follow what is indicated by the operation, but in $y = (x - h)^2$, the transformation is opposite to what the operation seems to indicate.

a) Explain why this might be so.

b) Describe the transformation you would use to graph $y = (2x)^2$.

13. A parabola $y = ax^2 + k$ passes through the points $(-1, 3)$ and $(3, -13)$. Find the values of a and k.

14. Compare the graphs of $y = (x - 2)^2$ and $y = (2 - x)^2$. Explain any similarities and differences.

15. Math Contest

a) Identify the similarities and differences in the graphs of $y = (x - 2)^2 + 5$ and $x = (y - 2)^2 + 5$.

b) Solve the second equation for y.