4.4 Graph $y = a(x - h)^2 + k$

The world's most important industry competition for fireworks manufacturers is L'International des Feux Loto-Québec. This event, also known as the Montréal Fireworks Festival, is held each summer in Montréal. The fireworks are synchronized to music that is also broadcast over a local radio station. Competing countries are judged on the synchronization, choice of music, and quality and originality of the visual display.

Paths of projectiles, such as rockets, balls, and fireworks, are often modelled using quadratic relations.

Investigate

Tools

- TI-83 Plus or TI-84 Plus graphing calculator
- grid paper

Technology Tip

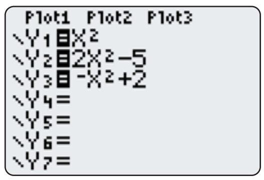
Turn off all stat plots by pressing 2nd Y= for [STAT PLOT], selecting **4:PlotsOff**, and then pressing ENTER.

How do the graphs of $y = a(x - h)^2 + k$ and $y = x^2$ compare?

1. First, clear any graphed equations, and ensure all stat plots are turned off.

2. Use a standard window.
- Press ZOOM and select **6:ZStandard.**
- View the window settings by pressing WINDOW.

3. Graph the equations $y = x^2$, $y = 2x^2 - 5$, and $y = -x^2 + 2$.



4. a) Sketch all three graphs on the same set of axes.

b) Label the coordinates of the vertex and a second point on each parabola.

c) Describe the transformations.

d) Without using a graphing calculator, sketch the graph of $y = -2x^2 + 1$.

5. a) Clear all equations except $y = x^2$.

b) Graph the equation $y = (x - 2)^2 + 1$.

6. a) Sketch the two graphs on the same set of axes.
   b) Label the coordinates of the vertex and a second point on each parabola.
   c) Draw the axis of symmetry for each parabola. Label each axis of symmetry with its equation.
   d) Describe the transformations.

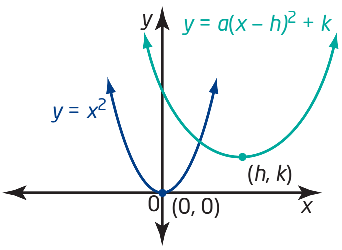
7. Without using a graphing calculator, sketch the graph of $y = (x - 1)^2 + 3$.

8. a) Repeat steps 5 and 6 for the equation $y = (x + 5)^2 - 4$.
   b) Without using a graphing calculator, sketch the graph of $y = (x + 4)^2 + 2$.

9. a) Repeat steps 5 and 6 for the equation $y = 2(x - 1)^2 - 5$.
   b) Without using a graphing calculator, sketch the graph of $y = -0.5(x + 2)^2 + 3$.

10. **Reflect** Write a summary of how to sketch a graph of a quadratic relation of the form $y = a(x - h)^2 + k$. Include a description of how to determine the coordinates of the vertex, the equation of the axis of symmetry, the values that x may take, and the values that y may take.

You can find the following from a quadratic relation of the form $y = a(x - h)^2 + k$:



- The vertex of the parabola is (h, k), representing a horizontal translation of h units and a vertical translation of k units relative to the graph of $y = x^2$.
- The axis of symmetry of the parabola is the vertical line through the vertex with equation $x = h$.
- a indicates the vertical stretch or compression factor relative to the graph of $y = x^2$.
  - If $a > 0$, the parabola opens upward, and the vertex is the minimum point on the graph.
  - If $a < 0$, the parabola opens downward, and the vertex is the maximum point on the graph.

Example 1 Sketch the Graph of $y = a(x - h)^2 + k$

a) Describe the properties of the parabola with equation $y = 2(x - 4)^2 - 3$.

b) Sketch a graph of the parabola and label it fully.

c) Describe the set of values that x may take.

d) Describe the set of values that y may take.

Solution

a) Compare $y = 2(x - 4)^2 - 3$ with $y = a(x - h)^2 - k$.

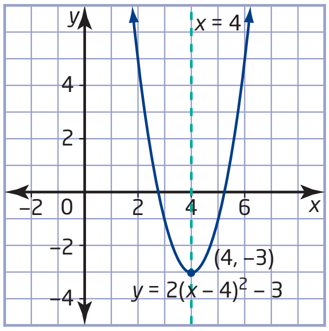
Since $a = 2$, the graph of $y = 2(x - 4)^2 - 3$ will be stretched by a factor of 2 compared to the graph of $y = x^2$.

The parabola will open upward, since a is positive.

The vertex is (h, k), or $(4, -3)$, and it is a minimum point.

The equation of the axis of symmetry is $x = 4$.

b)

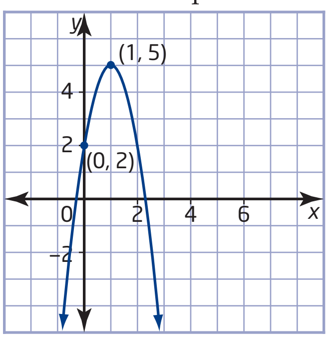


c) The graph shows that x may be any real number.

d) The graph shows that y may be any real number greater than or equal to -3, or $y \geq -3$.

Example 2 Write an Equation for a Graph

Determine an equation for the parabola shown.



Solution

The vertex is (1, 5), so $h = 1$ and $k = 5$.
The parabola opens downward, so a is negative.

Substitute the values for h and k into the equation $y = a(x - h)^2 + k$.

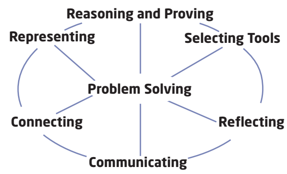
$y = a(x - 1)^2 + 5$

The parabola passes through the point (0, 2). Substitute $x = 0$ and $y = 2$ and solve for a.

$$2 = a(0 - 1)^2 + 5$$
$$2 = a(-1)^2 + 5$$
$$2 = a + 5$$
$$a = -3$$

An equation for the parabola is $y = -3(x - 1)^2 + 5$.

Example 3 Fireworks



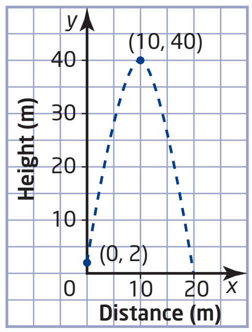
At a fireworks display, a firework is launched from a height of 2 m above the ground and reaches a maximum height of 40 m at a horizontal distance of 10 m.

a) Determine an equation to model the flight path of the firework.

b) The firework continues to travel an additional 1 m horizontally, after it reaches its maximum height, before it explodes. What is its height when it explodes?

c) At what other horizontal distance is the firework at the same height as in part b)?

Solution



a) Sketch a graph of the situation.

The launch height of 2 m above the ground represents the point (0, 2).

The maximum height is 40 m at a horizontal distance of 10 m, so the vertex is (10, 40). Substitute $h = 10$ and $k = 40$ into the equation $y = a(x - h)^2 + k$.

$y = a(x - 10)^2 + 40$

Substitute $x = 0$ and $y = 2$ and solve for a.

$$2 = a(0 - 10)^2 + 40$$
$$2 = a(-10)^2 + 40$$
$$2 = 100a + 40$$
$$-38 = 100a$$
$$a = -\frac{38}{100}$$
$$a = -0.38$$

An equation that models the flight path of the firework is $y = -0.38(x - 10)^2 + 40$, where x is the horizontal distance travelled, in metres, after the firework is launched and y is the height, in metres, above the ground.

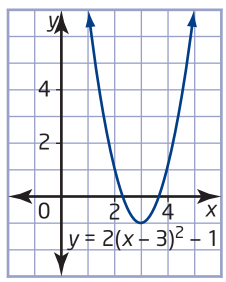
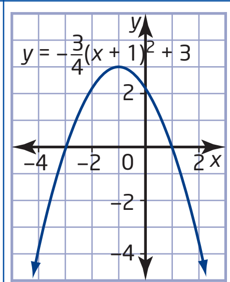
b) The firework travels an additional 1 m horizontally after its maximum height of 40 m at $x = 10$, so $x = 11$. Substitute $x = 11$ into $y = -0.38(x - 10)^2 + 40$.

$$\begin{aligned} y &= -0.38(11 - 10)^2 + 40 \\ &= -0.38(1)^2 + 40 \\ &= 39.62 \end{aligned}$$

The firework exploded at a height of 39.62 m.

c) Due to the symmetric property of a parabola, the firework is at the same height 1 m before the maximum point, or at a horizontal distance of 9 m.

Key Concepts

Property	$y = a(x - h)^2 + k$	$y = 2(x - 3)^2 - 1$	$y = -\frac{3}{4}(x + 1)^2 + 3$
Vertex	(h, k)	$(3, -1)$	$(-1, 3)$
Axis of symmetry	$x = h$	$x = 3$	$x = -1$
Stretch or compression factor relative to $y = x^2$	a	2	$-\frac{3}{4}$
Direction of opening	If $a > 0$, the parabola opens upward. The vertex is a minimum point. If $a < 0$, the parabola opens downward. The vertex is a maximum point.	Upward. $(3, -1)$ is a minimum point.	Downward. $(-1, 3)$ is a maximum point.
Graph	Parabola		
Values x may take	Any real number. Also depends on the situation.	Set of real numbers.	Set of real numbers.
Values y may take	If $a > 0$, then $y \geq k$. If $a < 0$, then $y \leq k$. Also depends on the situation.	$y \geq -1$	$y \leq 3$

Communicate Your Understanding

C1 Why is the vertical line through the vertex called the axis of symmetry? Illustrate with an example.

C2 When describing the transformation from $y = x^2$ to $y = 2x^2$, you say that it has been stretched vertically by a factor of 2, instead of compressed horizontally. Explain why vertical stretches are used in descriptions.

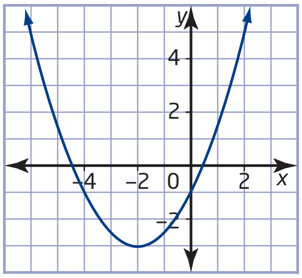
C3 Which equation is correct for the graph shown? Explain your reasoning.

A $y = (x + 2)^2 - 3$

B $y = \frac{1}{3}(x + 2)^2 - 3$

C $y = \frac{1}{2}(x + 2)^2 - 3$

D $y = -2(x + 2)^2 - 3$



Practise

For help with questions 1 and 2, see Example 1.

1. Copy and complete the table for each parabola. Replace the heading for the second column with the equation for the parabola.

Property	$y = a(x - h)^2 + k$
Vertex	
Axis of symmetry	
Stretch or compression factor relative to $y = x^2$	
Direction of opening	
Values x may take	
Values y may take	

a) $y = (x - 4)^2$

b) $y = (x - 2)^2 - 4$

c) $y = (x + 3)^2 - 2$

d) $y = \frac{1}{2}(x + 1)^2 + 5$

e) $y = (x - 7)^2 - 3$

f) $y = -(x - 1)^2 + 7$

g) $y = 2(x - 4)^2 - 5$

h) $y = -3(x + 4)^2 - 2$

2. Sketch each parabola in question 1.

For help with questions 3 to 7, see Example 2.

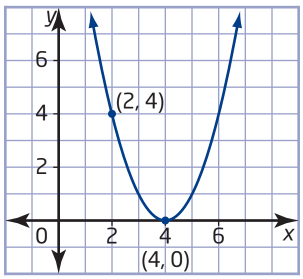
3. Write an equation for the parabola with vertex at (2, 3), opening upward, and with no vertical stretch.

4. Write an equation for the parabola with vertex at (−3, 0), opening downward, and with a vertical stretch of factor 2.

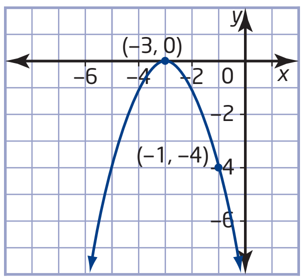
5. Write an equation for the parabola with vertex at (4, −1), opening upward, and with a vertical compression of factor 0.3.

6. Write an equation for each parabola.

a)

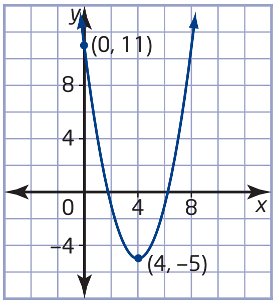


b)

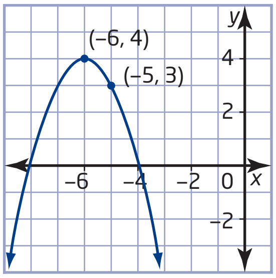


7. Write an equation for each parabola.

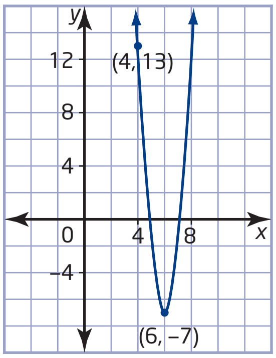
a)



b)



c)



Connect and Apply

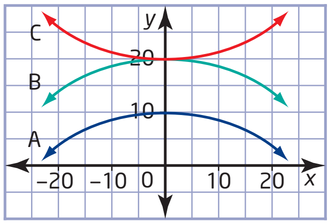
8. The graph of $y = x^2$ is stretched vertically by a factor of 3 and then translated 2 units to the left and 1 unit down. Sketch the parabola and write its equation.

9. The graph of $y = x^2$ is reflected in the x-axis, compressed vertically by a factor of $\frac{1}{2}$, and then translated 2 units upward. Sketch the parabola and write its equation.

10. a) Find an equation for the parabola with vertex (1, 4) that passes through the point (3, 8).

b) Find an equation for the parabola with vertex (−2, 5) and y-intercept 1.

11. A stadium roof has a cross section in the shape of a parabolic arch with equation $y = -\frac{1}{45}x^2 + 20$. Which graph represents the arch? Justify your reasoning.



For help with questions 12 and 13, see Example 3.

12. The path of a soccer ball is modelled by the relation $h = -\frac{1}{16}(d - 28)^2 + 49$, where d is the horizontal distance, in metres, after it was kicked, and h is the height, in metres, above the ground.

a) Sketch the path of the soccer ball.

b) What is the maximum height of the ball?

c) What is the horizontal distance when this occurs?

d) What is the height of the ball at a horizontal distance of 20 m?

e) Find another horizontal distance where the height is the same as in part d).

13. A baseball is batted at a height of 1 m above the ground and reaches a maximum height of 33 m at a horizontal distance of 4 m.

a) Determine an equation to model the path of the baseball.

b) What is the height of the baseball once it has travelled a horizontal distance of 6 m?

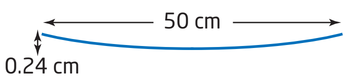
c) At what other horizontal distance is the baseball at the same height as in part b)?

14. The flight path of a firework is modelled by the relation $h = -5(t - 5)^2 + 127$, where h is the height, in metres, of the firework above the ground and t is the time, in seconds, since the firework was fired.

a) What is the maximum height reached by the firework? How many seconds after it was fired does the firework reach this height?

b) How high was the firework above the ground when it was fired?

15. Parabolic mirrors are used in telescopes to help magnify the image. The cross section of a parabolic mirror is shown.



a) Sketch a graph to represent the cross section of the mirror, placing the vertex at $(0, -0.24)$.

b) Write an equation to represent the cross section of the mirror. Describe the values of x for which your equation is valid.

c) Move the mirror to a different location on the same set of axes. Write a different equation to represent the cross section of the mirror. Describe the values of x for which your equation is valid.

Did You Know?

The Olympic Torch is lit by placing it in a parabolic mirror that concentrates rays of sunlight.

16. Chapter Problem Use Technology A city opened a new landfill site in 2000. In Section 4.1, question 6, you created a table of values showing the total mass of garbage in the landfill for each year from 2000 to 2007.

a) Use a graphing calculator to create a scatter plot of the data and draw a curve of best fit.

- With the scatter plot displayed, press STAT, cursor over to display the **CALC** menu, and select **5:QuadReg**.
- Press VARS, and cursor over to display the **Y-VARS** menu. Select **1:Function** and then select **1:Y1**.
- Press ENTER to get the **QuadReg** screen, and press GRAPH.

b) Use the Minimum operation of a graphing calculator to find the coordinates of the vertex.

- Adjust the window settings so you can view the vertex of the parabola.
- Press 2nd [CALC] to display the **CALCULATE** menu, and select **3:minimum**.
- Move the cursor to the left of the vertex and press ENTER.
- Move the cursor to the right of the vertex and press ENTER.
- Press ENTER again.

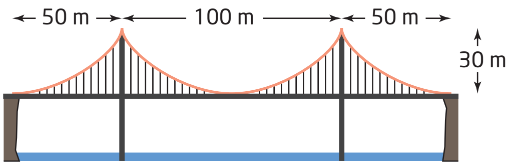
c) Sketch the graph of the curve of best fit. Label the coordinates of the vertex and one other point on your sketch. Use these points to determine an equation for the curve of best fit in the form $y = a(x - h)^2 + k$.

d) Describe the set of values that x may take and the set of values that y may take.

Making Connections

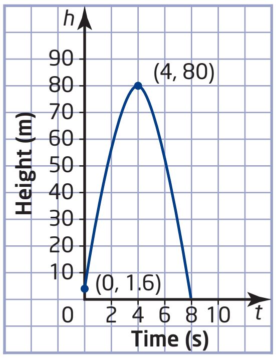
Refer to the Technology Appendix for help with creating a scatter plot on a TI-83 Plus or TI-84 Plus graphing calculator.

17. The cables of a suspension bridge form parabolas. If the minimum point of the centre cable is placed at the origin, determine an equation for each parabola. Describe the values of x for which each equation is valid.



Achievement Check

18. The graph shows the path of a rocket fired from the deck of a barge in Lake Ontario at a Canada Day fireworks display. It is a parabola, where h is the height, in metres, of the rocket above the water and t is the time, in seconds.



a) What is the maximum height reached by the rocket? Justify your answer.

b) When did the rocket reach its maximum height? Justify your answer.

c) How high was the rocket above the water when it was set off? Explain your answer.

d) Find an equation to describe the flight of the rocket.

e) After how long did the rocket fall into the water? Explain your answer.

Extend

19. A parabola has equation $y = 2(x - 4)^2 - 1$. Write an equation for the parabola after each transformation.

a) a reflection in the x-axis

b) a translation of 4 units to the left

c) a reflection in the x-axis, followed by a translation of 3 units upward

d) a reflection in the y-axis

20. a) In Chapter 2, you developed the equation of a circle centred at the origin. What is the equation of each circle?

- radius 5, centred at (0, 3)
- radius 7, centred at (6, 1)
- radius 8, centred at (−3, 5)
- radius r, centred at (h, k)

b) What do the equation of a circle and the equation of a parabola have in common?

21. Math Contest A locus is a set of points that satisfy a specific condition. For example, a circle is a set of points that are equidistant from a fixed point (the centre). Find an equation for the locus of points that is equidistant from the point (3, 2) and the line $y = -5$.

22. Math Contest Given that $a + b = 21$ and $\frac{1}{a} + \frac{1}{b} = \frac{7}{18}$, the value of ab is

A 18

B 36

C 54

D 72

E 90