5.2 Special Products

The patterns in tile floors, brick patios, and quilts often repeat. Some quilts are made from square pattern blocks of material that are stitched together. The square pattern block shown starts with a square that is enlarged with the addition of rectangles to each of its dimensions until the desired square dimensions are reached. If the side length of the square pattern block is represented by $x + 3$, what is an expression for its area?

Investigate

How can you use patterns to find special products?

Tools

Optional

- algebra tiles

A: Squaring Binomials

Method 1: Use Pencil and Paper

1. Use algebra tiles or a diagram to square the binomial.
$(x + 3)^2 = (x + 3)(x + 3)$

2. Expand and simplify.

a) $(x + 3)^2$ **b)** $(x + 2)^2$ **c)** $(x - 6)^2$
d) $(x - 4)^2$ **e)** $(2x + 5)^2$ **f)** $(3x - 1)^2$
g) $(2x - 5y)^2$ **h)** $(4x + 7y)^2$

3. Consider each simplified expansion from step 2.

a) How is the first term in each trinomial related to the first term in each binomial?

b) How is the last term in each trinomial related to the last term in each binomial?

c) How is the middle term in each trinomial related to the terms in the binomial?

4. **Reflect** Write a rule for expanding and simplifying $(a + b)^2$ or $(a - b)^2$.

5. Use your rule to square each binomial.

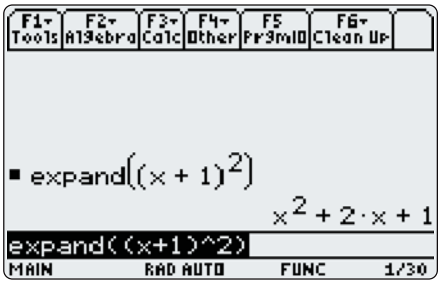
a) $(5x + 3y)^2$ **b)** $(7c - 4d)^2$

Method 2: Use a Computer Algebra System (CAS)

Tools
- TI-89 calculator

1. Clear the calculator's memory by selecting **2:NewProb** from the **Clean Up** menu.

2. Use the **Expand** function on each square of a binomial. Record the results.



   a) $(x + 1)^2$ **b)** $(x + 2)^2$
   c) $(x + 3)^2$ **d)** $(x + 4)^2$
   e) $(x + 5)^2$

3. Compare the individual terms of the expansion to the binomial. Describe any patterns you notice, including any for the signs of the terms in the trinomial.

4. Use a CAS to expand each square of a binomial. Record the results.
   a) $(2x + 2)^2$ **b)** $(2x + 3)^2$
   c) $(2x - 4)^2$ **d)** $(2x - 5)^2$

5. Compare the individual terms of the expansion to the binomial. Describe any patterns you notice. How has your description of patterns changed compared to step 3?

6. Use a CAS to expand each square of a binomial. Record the results.
   a) $(3a + 2)^2$ **b)** $(5m - 3)^2$
   c) $(4 + 2b)^2$ **d)** $(7 - 3z)^2$
   e) $(2x + 3y)^2$

7. Compare the individual terms of the expansion to the binomial. Have the patterns you described in step 5 changed?

8. **Reflect** Write a rule for expanding each square of a binomial.
   a) $(a + b)^2$
   b) $(a - b)^2$

B: Product of a Sum and a Difference of Two Terms

Method 1: Use Pencil and Paper

1. Expand and simplify.
   a) $(x + 3)(x - 3)$
   b) $(2y + 5)(2y - 5)$
   c) $(x - 4)(x + 4)$
   d) $(3k - 7)(3k + 7)$

2. How are the two binomials in each multiplication in step 1 alike? How are they different?

3. Consider each simplified expansion from step 1.
   a) How is the first term related to the first terms of the two binomials?
   b) How is the last term related to the last terms of the two binomials?
   c) Explain why there are only two terms in the simplified expansion.

4. **Reflect** Write a rule for expanding and simplifying $(a + b)(a - b)$. Does your rule apply to $(a - b)(a + b)$? Explain.

5. Use your rule to find the product of each sum and difference of two terms.
   a) $(2x + 3y)(2x - 3y)$
   b) $(5m + 7)(5m - 7)$

Tools
- TI-89 calculator

Method 2: Use a CAS

1. Clear the calculator's memory by selecting **2:NewProb** from the **Clean Up** menu.

2. Use the **Expand** function on each binomial product. Record the results.
   a) $(x + 2)(x - 2)$
   b) $(x - 3)(x + 3)$
   c) $(2x - 1)(2x + 1)$
   d) $(3x + 4)(3x - 4)$
   e) $(2x + 3y)(2x - 3y)$
   f) $(3m - 4n)(3m + 4n)$

3. How are the two binomials in each multiplication in step 2 alike? How are they different?

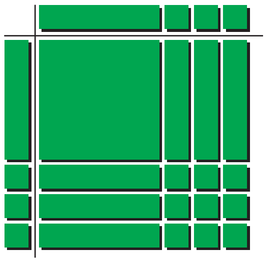
4. Consider each expansion from step 2.
   a) How is the first term related to the first terms of the two binomials?
   b) How is the last term related to the last terms of the two binomials?
   c) Explain why there are only two terms in the simplified expansion.

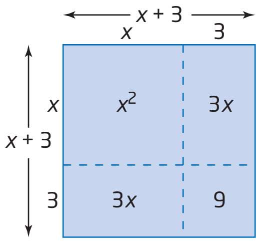
5. **Reflect** Write a rule for expanding and simplifying $(a + b)(a - b)$. Does your rule apply to $(a - b)(a + b)$? Explain.

While you can always use the distributive property to find the product of two binomials, some pairs of binomials result in special products that follow specific patterns.

Squaring Binomials

You can visualize squaring a binomial using algebra tiles or a diagram. Surrounding the x^2 region, there are *two* regions of $3x$, which combine to give $6x$, and a region defined by 3×3, or 9, giving $x^2 + 6x + 9$.





$$(x + 3)^2 = (x + 3)(x + 3)$$
$$= x^2 + 3x + 3x + 3^2$$
$$= x^2 + 2(3x) + 9$$
$$= x^2 + 6x + 9$$

Alternatively, you can use the patterns for squaring a binomial. Square the first term, find twice the product of the terms, and square the last term. The middle term is added when you square a sum and subtracted when you square a difference.

$(a + b)^2 = a^2 + 2ab + b^2$ or $(a - b)^2 = a^2 - 2ab + b^2$

The resulting products are called **perfect square trinomials**.

Product of a Sum and a Difference

When you multiply the sum and the difference of two terms, there are only two terms in the simplified expansion. There is no middle term because the other two products have a sum of zero.

$$(x + 3)(x - 3) = x^2 - 3x + 3x - 3^2$$
$$= x^2 - 9$$

To find the product of the sum and the difference of two terms, use the pattern. Find the product of the first terms and the product of the second terms.

$(a + b)(a - b) = a^2 - b^2$

I can change the order of the binomial factors and get the same result.
$(a - b)(a + b) = a^2 - b^2$
This happens because I can multiply two numbers or expressions in either order.

The resulting product, $a^2 - b^2$, is called a **difference of squares**.

perfect square trinomial

- a trinomial of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ that is the result of squaring a binomial

Literacy Connections

When you change the operation between the two terms of a binomial, the two forms are called conjugates. $x + 3$ and $x - 3$ are conjugates.

difference of squares

- an expression of the form $a^2 - b^2$ that involves the subtraction of two squares

Example 1 Apply Special Product Patterns

Expand and simplify.

a) $(x + 4)^2$ **b)** $(k - 5)^2$ **c)** $(3y + 7x)^2$

d) $(q - 11)(q + 11)$ **e)** $(4m + 3n)(4m - 3n)$

Solution

a)
$$\begin{aligned}(a + b)^2 &= a^2 + 2ab + b^2\\ (x + 4)^2 &= (x)^2 + 2(x)(4) + (4)^2\\ &= x^2 + 8x + 16\end{aligned}$$

Use the appropriate pattern for squaring a binomial.

b)
$$\begin{aligned}(a - b)^2 &= a^2 - 2ab + b^2\\ (k - 5)^2 &= (k)^2 - 2(k)(5) + (5)^2\\ &= k^2 - 10k + 25\end{aligned}$$

Use the appropriate pattern for squaring a binomial.

c)
$$\begin{aligned}(a + b)^2 &= a^2 + 2ab + b^2\\ (3y + 7x)^2 &= (3y)^2 + 2(3y)(7x) + (7x)^2\\ &= 9y^2 + 42xy + 49x^2\end{aligned}$$

Use the appropriate pattern for squaring a binomial.

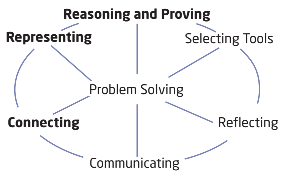
d)
$$\begin{aligned}(a - b)(a + b) &= a^2 - b^2\\ (q - 11)(q + 11) &= (q)^2 - (11)^2\\ &= q^2 - 121\end{aligned}$$

Use the pattern for the product of a sum and a difference.

e)
$$\begin{aligned}(a + b)(a - b) &= a^2 - b^2\\ (4m + 3n)(4m - 3n) &= (4m)^2 - (3n)^2\\ &= 16m^2 - 9n^2\end{aligned}$$

Use the pattern for the product of a sum and a difference.

Example 2 Helicopter Pad



The radius of a circular helicopter landing pad is increased by 3 m.

a) Find a simplified expression for the area of the new circle.

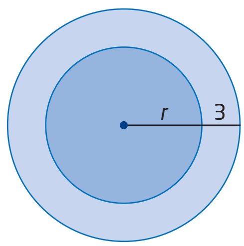
b) Find a simplified expression for the increase in area.

Solution

a) Area of the original circle $= \pi r^2$

$$\begin{aligned}\text{Area of the larger circle} &= \pi(r + 3)^2\\ &= \pi[(r)^2 + 2(r)(3) + (3)^2]\\ &= \pi(r^2 + 6r + 9)\\ &= \pi r^2 + 6\pi r + 9\pi\end{aligned}$$

I can use the appropriate pattern for squaring a binomial.



b)
$$\begin{aligned}\text{Increase in area} &= (\text{area of larger circle}) - (\text{area of original circle})\\ &= (\pi r^2 + 6\pi r + 9\pi) - (\pi r^2)\\ &= 6\pi r + 9\pi\end{aligned}$$

Key Concepts

- When squaring a binomial, you add the two equal middle terms after expansion.
  $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$
- When you multiply the sum and the difference of two terms, the two middle terms are opposites, so they add to zero.
  $(a + b)(a - b) = a^2 - b^2$

Communicate Your Understanding

C1 Explain when the middle term is positive and when it is negative when you are squaring a binomial.

C2 Describe the steps you would use to expand and simplify each product.

a) $(x - 2)^2$ **b)** $(x - 2)(x + 2)$

Practise

1. Draw a diagram to illustrate each product.

a) $(x + 5)^2$ **b)** $(x + 6)^2$
c) $(x + a)^2$ **d)** $(ax + b)^2$

For help with questions 2 to 6, see Example 1.

2. Expand and simplify.

a) $(x + 5)^2$ **b)** $(y + 4)^2$ **c)** $(w + 6)^2$
d) $(k + 7)^2$ **e)** $(m + 11)^2$ **f)** $(c + 10)^2$
g) $(g + 9)^2$ **h)** $(x + 20)^2$

3. Expand and simplify.

a) $(x - 5)^2$ **b)** $(z - 3)^2$ **c)** $(x - 9)^2$
d) $(c - 1)^2$ **e)** $(v - 12)^2$ **f)** $(b - 100)^2$
g) $(n - 2)^2$ **h)** $(m - 6)^2$

4. Expand and simplify.

a) $(x + 3y)^2$ **b)** $(2x - y)^2$ **c)** $(5c + 2d)^2$
d) $(3a - 4b)^2$ **e)** $(9k + 2m)^2$ **f)** $(4u - 5v)^2$

5. Expand and simplify.

a) $(v + 1)(v - 1)$ **b)** $(a - 1)(a + 1)$
c) $(y + 5)(y - 5)$ **d)** $(x - 7)(x + 7)$
e) $(e - 9)(e + 9)$ **f)** $(z + 6)(z - 6)$
g) $(x + 12)(x - 12)$ **h)** $(y - 3)(y + 3)$

6. Expand and simplify.

a) $(w - v)(w + v)$ **b)** $(3m - n)(3m + n)$
c) $(y + 6x)(y - 6x)$ **d)** $(3x + 4y)(3x - 4y)$
e) $(7g - 3h)(7g + 3h)$
f) $(9x - 8y)(9x + 8y)$

Connect and Apply

7. Expand and simplify. Verify your answers using one of three methods:

- Check that substituting $x = 2$ into the original expression and the simplified expansion yields the same answer.
- Check that graphing both the original expression and the simplified expansion using a graphing calculator yields only one graph.
- Check that using a CAS to expand the original expression yields the same answer.

a) $(x + 4)(x - 4)$ **b)** $(x - 8)^2$
c) $(x + 8)^2$ **d)** $(x - 10)(x + 10)$
e) $(x + 11)(x - 11)$ **f)** $(x + 12)^2$
g) $(x - 7)^2$ **h)** $(x - 30)(x + 30)$

For help with questions 8 and 9, see Example 2.

8. The radius, r, of a circle has been increased by k. Both r and k are measured in the same units. Write a formula for the area of the new circle. Expand and simplify.

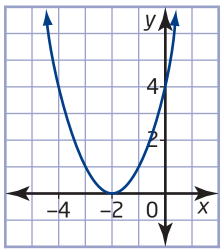
9. Each dimension of a square playground is increased by 5 m.

a) Draw a diagram of the situation.

b) Find a simplified algebraic expression for the area of the new playground.

c) Find a simplified algebraic expression for the increase in area.

10. A parabola has equation $y = (x + 2)^2$.



a) Identify the coordinates of the vertex.

b) Expand and simplify the equation.

c) Verify that the coordinates of the vertex satisfy the equation from part b).

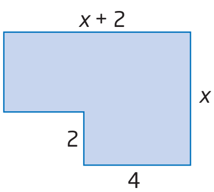
11. A square has side length $3x$. One dimension is increased by $2y$ and the other is decreased by $2y$.

a) Find an algebraic expression for the area of the resulting rectangle. Expand.

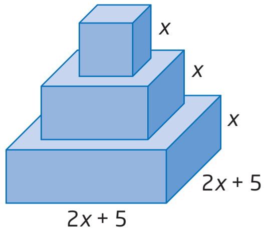
b) Find an algebraic expression for the change in area. Expand.

c) Calculate the area of the rectangle and change in area if x represents 8 cm and y represents 5 cm.

12. Use two methods to determine an algebraic expression to represent the area of the figure. Verify that they are equivalent expressions.



13. **Chapter Problem** A pedestal for trophies is to be made from three layers, each in the shape of a square-based prism. All three layers are the same height, x, in centimetres, but each base length is 3 cm less than that of the layer immediately below.



a) Write algebraic expressions for the side and top surface areas of each prism used to make the pedestal.

b) Write an algebraic expression for the exposed top surface area of the bottom layer of the pedestal. Expand and simplify.

c) Write an algebraic expression for the exposed top surface area of the middle layer of the pedestal. Expand and simplify.

14. An interesting way to multiply 19×21 is

$$\begin{aligned} &(20 - 1)(20 + 1) \\ &= 20^2 - 1 \\ &= 400 - 1 \\ &= 399 \end{aligned}$$

Use this method to calculate each product.

a) 31×29

b) 59×61

c) 99×101

d) 71×69

15. Explain how to adapt the method in question 14 to multiply 32×28. Then, use your method to multiply each product.

a) 76×84

b) 35×25

c) 104×96

d) 77×83

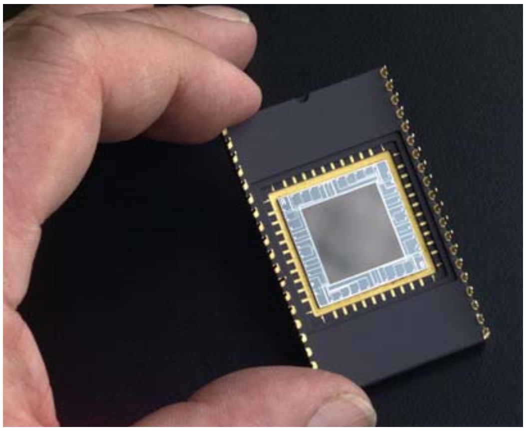
16. A stone is dropped from a height of 10 m. Its height as it falls can be approximated by the relation $h = -5t^2 + 10$, where t is the time, in seconds, and h is the height, in metres.

a) A delay of 3 s would cause the graph to shift 3 units to the right. Sketch a graph of this relation and of the relation after a 3-s delay.

b) Rewrite the relation to represent a delay of 3 s.

c) Expand the new relation and simplify.

17. Instead of film, a digital camera has an image sensor that converts light into electrical charges. The image sensor in most digital cameras is a charge coupled device (CCD). The CCD for an 8-megapixel digital camera measures 3264 pixels by 2448 pixels. A new model of CCD increases each dimension by x pixels.



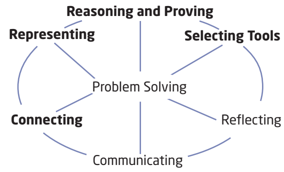
a) Write an expression for the number of pixels in the new CCD, and expand.

b) If x represents 1000 pixels, what is the resolution of the new CCD, in megapixels?

Literacy Connections

Mega is a prefix in the SI system of units for 10^6, or 1 000 000. For example, 1 megapixel equals one million pixels.

Achievement Check



18. a) Use algebra tiles or a diagram to expand $(a + b)^2$.

b) Use a diagram to expand $(a + b + c)^2$.

c) The length and width of a rectangle are represented by $x + 2$ and $9 - 4x$. If x must be an integer, what are the possible values for the area of the rectangle?

Extend

19. Expand and simplify. Use a CAS to verify your answers.

a) $(x - 2)^4$

b) $(2x + 3)(x - 5)(4x + 7)$

c) $(2x^2 + 5x + 3)^2$

d) $(5x - 2)^3$

20. The kinetic energy, E, in joules, of a moving object is given by the formula $E = \frac{1}{2}mv^2$, where m is the mass, in kilograms, of the object, and v is its speed, in metres per second.

a) Find an algebraic expression for the difference in kinetic energy of two objects with a difference in speed of 5 m/s.

b) Find an algebraic expression for the difference in kinetic energy of two objects with a difference in speed of x metres per second.

c) Expand each expression and simplify.

21. The sum of the cubes of the first n natural numbers can be found using the formula

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}.$$

a) Verify that this is true for the first five natural numbers.

b) Show that the right side of the formula can also be expressed in the form $\left[\frac{n(n + 1)}{2}\right]^2$.