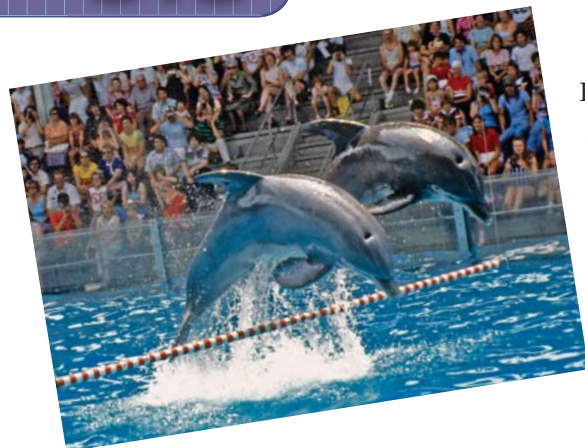


5.3

Common Factors



During a performance at a sea-life park, a dolphin jumps out of the water. Its height, h , in metres, above the water after t seconds can be approximated by the relation $h = 10t - 5t^2$. This relation can also be written as $h = 5t(2 - t)$, because the terms in the polynomial $10t - 5t^2$ have a common factor of $5t$.

Tools

■ algebra tiles

Investigate A

How can you use a model to find factors of a polynomial?

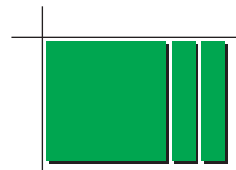
1. To factor $2x + 4$, use algebra tiles to create a rectangular area whose length and width represent the factors of the polynomial.
 - a) Arrange two x -tiles and four unit tiles to form a rectangle with area $2x + 4$. Then, place tiles along the left side and top to find the length and width of the rectangle. One dimension has been done for you.



- b) Write an equation for the area as a product of the length and width.

2. Repeat step 1 for $6x + 18$. How many different rectangles can you find?

3. Use algebra tiles to find the factors of $x^2 + 2x$. Express the area as a product of the length and width.



4. Use algebra tiles to factor $2x^2 + 4x$. How many different rectangles can you find? Write an area statement for each one.

5. Use algebra tiles to factor each expression, if possible. If it is not possible, explain why.

- | | | |
|----------------|--------------|-----------------|
| a) $3x + 3$ | b) $4x + 10$ | c) $x^2 + 4x$ |
| d) $2x^2 + 6x$ | e) $2x + 5$ | f) $4x^2 + 10x$ |

6. **Reflect** Explain how you can express a polynomial as a product of factors.

Investigate B

How can you use the greatest common factor (GCF) to factor a polynomial?

Method 1: Use Pencil and Paper

- Find the GCF for each set of numbers by first expressing each number as a product of prime factors.
 - 12 and 8
 - 15 and 25
 - 4, 10, and 6
 - 6, 18, and 24
- Find the GCF of 12 and 9.
 - Write each number as a product of two factors, where the first factor is the GCF. What operation did you use to obtain the second factor?
- Find the GCF of each pair of terms.
 - 7^3 and 7^2
 - 5^6 and 5^4
 - x^2 and x
 - x^3 and x^4
- Find the GCF of x^6 and x^4 .
 - Write each term as a product of two factors, where the first factor is the GCF. What operation did you use to obtain the second factor?
- Find the GCF of the polynomial $2x^2 + 4x$ by first expressing each term as a product.
 $2x^2 = 2 \times x \times x$
 $4x = 2 \times 2 \times x$
Multiply the common factors to calculate the GCF.
 - What is the GCF of $2x^2 + 4x$?
 - Rewrite the polynomial as the sum of products. Express each term as a product of two factors, where the first factor is the GCF. Use division to determine the second factor of each term.
 - Write the polynomial as a product of two factors, where the first factor is the GCF. What polynomial is the second factor?
 - Verify your factors from part c) by expanding.
- Repeat step 5 for each polynomial.
 - $3x^2 + 6x$
 - $2x^2 + 8x$
 - $4y + 10y^2$
 - $7y^3 + 14y^2$
- Reflect** Explain how to factor a polynomial using the GCF.

Method 2: Use a Computer Algebra System (CAS)

- Clear the calculator's memory by selecting **2:NewProb** from the **Clean Up** menu.
- Factor each expression. Press **F2**. Select **2:factor(**. Type the expression, and then press **)** **ENTER**. Record the results.
 - $2x + 2$
 - $2x + 4$
 - $2x + 6$
 - $2x + 8$
 - $2x + 10$
- Consider the pairs of factors from step 2. How are they the same? How are they different? Provide a reason for your answers.
- Use a CAS to factor each polynomial. Record the results.
 - $3x^2 + 6x + 9$
 - $3x^2 + 12x + 15$
- Look at the coefficients of the terms of each polynomial in step 4. What is the GCF?
 - Look at the terms in the second factor from step 4. Since the first factor is the GCF of the polynomial, what operation provides the coefficients of the terms in the second factor?

- Use a CAS to factor $6x^2 + 9x + 24$. Record the result.



- What is the GCF of the polynomial $6x^2 + 9x + 24$?
 - Use a CAS to divide the GCF into each term of $6x^2 + 9x + 24$, as shown. Does the result match the second factor from part a)?
- Use a CAS to factor each polynomial. Record the results.
 - $2x^2 + 4x$
 - $2x^2 + 8x$
 - Look at the coefficients of the terms of each polynomial in step 7. What is the GCF? Look at the variable parts. What is the GCF?
 - The first factor of each result from step 7 is the GCF of the polynomial. How is it related to your answers for part a)?
 - Look at the terms in the second factor from step 7. Since the first factor is the GCF of the polynomial, what operation provides the terms in the second factor?
 - Use a CAS to factor $6x^2 + 15x$. Record the result.
 - What is the GCF of the polynomial $6x^2 + 15x$?
 - Use a CAS to divide the GCF into each term of $6x^2 + 15x$. Does the result match the second factor from part a)?
 - Reflect** Explain how to factor a polynomial using the GCF.

A polynomial is factored when it is written as a product of two or more polynomials. Factoring a polynomial is the reverse process of expanding. To factor a polynomial:

- Find the GCF of the terms.
- Write the GCF as the first factor outside a set of brackets.
- Divide each term by the GCF, writing the result inside the brackets.

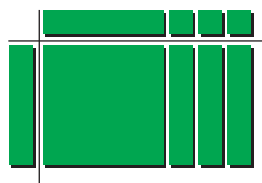
Example 1 Use a Model

Use algebra tiles to factor $x^2 + 3x$.



Solution

The polynomial $x^2 + 3x$ can be represented by a rectangle with area $x^2 + 3x$. The width of the rectangle is x and the length is $x + 3$. The dimensions of the rectangle are the factors of the polynomial.



$$x^2 + 3x = x(x + 3)$$

Example 2 Monomial Common Factor

Factor fully, if possible.

- a) $6x + 3$ b) $8x^2 - 7x$
 c) $25k^6 + 15k^4$ d) $21c^4d^3 - 28c^2d^5 + 7cd^3$
 e) $5x^5y^3 + 7w^5z^2$

Solution

- a) The GCF of the coefficients, 6 and 3, is 3.
 There is no GCF of the variable parts.
 Therefore, the GCF of the polynomial is 3.
 Divide each term by 3.

$$\begin{aligned} 6x + 3 &= 3\left(\frac{6x}{3} + \frac{3}{3}\right) \\ &= 3(2x + 1) \end{aligned}$$

- b) There is no common factor of the coefficients.
 The GCF of the variable parts, x^2 and x , is x .
 Therefore, the GCF of the polynomial is x .
 Divide each term by x .

$$\begin{aligned} 8x^2 - 7x &= x\left(\frac{8x^2}{x} - \frac{7x}{x}\right) \\ &= x(8x - 7) \end{aligned}$$

- c) The GCF of the coefficients, 25 and 15, is 5.
The GCF of the variable parts, k^6 and k^4 , is k^4 .
Therefore, the GCF of the polynomial is $5k^4$.
Divide each term by $5k^4$.

$$25k^6 + 15k^4 = 5k^4 \left(\frac{25k^6}{5k^4} + \frac{15k^4}{5k^4} \right)$$

$$= 5k^4(5k^2 + 3)$$

Apply the exponent laws.

When I compare the exponents, the GCF of the variable parts is the variable with the least exponent.

- d) The GCF of the coefficients, 21, -28 , and 7, is 7.

The GCF of the variable parts, c^4d^3 , c^2d^5 and cd^3 , is cd^3 .

Therefore, the GCF of the polynomial is $7cd^3$.

Divide each term by $7cd^3$ mentally.

$$21c^4d^3 - 28c^2d^5 + 7cd^3 = 7cd^3(3c^3 - 4cd^2 + 1)$$

I can check my answer by expanding.

$$7cd^3(3c^3 - 4cd^2 + 1)$$

$$= 21c^4d^3 - 28c^2d^5 + 7cd^3$$

- e) Since the GCF of the terms of the polynomial $5x^5y^3 + 7w^5z^2$ is 1, it is not factorable.

Example 3 Binomial Common Factor

Factor.

a) $3x(y + 1) + 7z(y + 1)$

b) $2x(x - 3) - 5(x - 3)$

Solution

- a) Think of $(y + 1)$ as one factor.

The GCF is the binomial $(y + 1)$.

Divide each term by $(y + 1)$ mentally.

$$3x(y + 1) + 7z(y + 1)$$

$$= (y + 1)(3x + 7z)$$

Write the GCF first. Collect the $(3x + 7z)$ in the second set of brackets.

- b) Think of $(x - 3)$ as one factor.

The GCF is the binomial $(x - 3)$.

Divide each term by $(x - 3)$ mentally.

$$2x(x - 3) - 5(x - 3)$$

$$= (x - 3)(2x - 5)$$

Often there is no common factor for all the terms in a polynomial, but some of the terms have a common factor. A process of factoring by grouping can sometimes be used with these polynomials. This process involves factoring groups of terms first, instead of factoring the entire polynomial.

Example 4 Factor by Grouping

Factor.

a) $ax + ay + 2x + 2y$

b) $9x^2 + 15x + 3x + 5$

Solution

- a) Group terms with a common factor. Factor the GCF from each grouping. Then, remove the binomial common factor.

$$\begin{array}{l} ax + ay + 2x + 2y \\ = (ax + ay) + (2x + 2y) \\ = a(x + y) + 2(x + y) \\ = (x + y)(a + 2) \end{array} \quad \text{or} \quad \begin{array}{l} ax + ay + 2x + 2y \\ = (ax + 2x) + (ay + 2y) \\ = x(a + 2) + y(a + 2) \\ = (a + 2)(x + y) \end{array}$$

b) $9x^2 + 15x + 3x + 5$ or $9x^2 + 15x + 3x + 5$

$$\begin{array}{l} = (9x^2 + 15x) + (3x + 5) \\ = 3x(3x + 5) + 1(3x + 5) \\ = (3x + 5)(3x + 1) \end{array} \quad \begin{array}{l} = (9x^2 + 3x) + (15x + 5) \\ = 3x(3x + 1) + 5(3x + 1) \\ = (3x + 1)(3x + 5) \end{array}$$

Key Concepts

- Factoring a polynomial is the opposite of expanding a polynomial.

— **Factoring** —→

$$x^2 + 3x = x(x + 3)$$

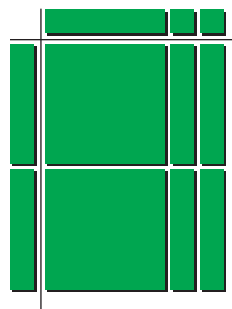
← **Expanding** —

- To find the GCF of a polynomial, find the GCF of the coefficients, and then find the GCF of the variable parts.
- To factor a polynomial, remove the GCF as the first factor, and then divide each term by the GCF to obtain the second factor.
 $8x^2y^3 - 12x^4y = 4x^2y(2y^2 - 3x^2)$
- For polynomials with more than one variable, the GCF of the variable parts is the product of the common bases with the least exponent.
The GCF of $2x^3y^4z^2 + 4x^2y^2z^3$ is $2x^2y^2z^2$.
- A common factor is not necessarily a monomial.
 $a(x + 2) + b(x + 2)$ has a binomial common factor of $(x + 2)$.
- To factor by grouping, factor groups of two terms with a common factor to produce a binomial common factor.

$$\begin{aligned} bx + 3x + by + 3y &= (bx + 3x) + (by + 3y) \\ &= x(b + 3) + y(b + 3) \\ &= (b + 3)(x + y) \end{aligned}$$

Communicate Your Understanding

- C1** Explain how the diagram illustrates the factoring of a polynomial.
- C2** Each of the following is an example of a common error when factoring. Describe each error and make the appropriate correction.
- $35x^2 - 5x = 5x(7x - 0)$
 - $4y^3 + 7y^2 = 4y^2(y + 7)$
 - $16k^3m^2 - 8k^2m = 4k^2m(4km - 2)$
 - $9a^3b^5 + 6a^2b^4 = 3ab(3a^2b^4 + 2ab^3)$
- C3** Describe how you would factor each polynomial.
- $3c(d - 5) - 8(d - 5)$
 - $10x^2 - 14xy - 15x + 21y$



Practise

- Find the GCF of each pair of terms.
 - $2x$ and $3x$
 - $6ab$ and $-8ac$
 - x^2 and x^3
 - k^4 and k^7
 - $3m^2$ and $5m$
 - $-12y^2$ and $-15y^4$

For help with question 2, see Example 1.

- Use algebra tiles or a diagram to illustrate the factoring of each polynomial.
 - $x^2 + 5x$
 - $3x^2 + 6x$
 - $6x^2 + 4x$

For help with questions 3 and 4, see Example 2.

- Factor fully, if possible.
 - $15w + 25z$
 - $3a - 11b$
 - $17ca - 8cd$
 - $9y - 8y^3$
 - $12b^4 + 18b^2$
 - $4g^2 - 8g + 6$
 - $7h + 3m - 5k$
 - $2n^5 + 12n^4 - 6n^3$
- Factor fully, if possible.
 - $14x^2y + 16xy^3$
 - $10k^3m^2 - 6k^2m^2$
 - $8s^2y + 11t^3$
 - $66c^4de^2 - 22c^2de^2$
 - $7gh + 2mn - 13pq$
 - $5fg^2 - 25fg + 20f^2g$
 - $27r^2s^2 - 18r^3s^2 - 36rs^3$
 - $4n^2p^3 + 10n^4p^2 - 12n^3p^2$

For help with question 5, see Example 3.

- Factor, if possible.
 - $3x(x + 8) + 5(x + 8)$
 - $a(b + 1) + 9c(b + 1)$
 - $2y(x - 5) + 4(x + 5)$
 - $4s(r + u) - t(r + u)$

For help with question 6, see Example 4.

- Factor by grouping.
 - $mx + my + 2x + 2y$
 - $x^2 + 3x + 2x + 6$
 - $ay^2 + 3ay + 4y + 12$
 - $6x^2 + 9x - 2x - 3$
 - $16v^2 - 12v - 12v + 9$

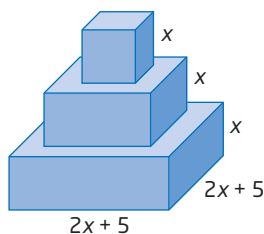
Connect and Apply

- Write a polynomial with two terms that has a GCF of 6.
 - Write a polynomial with three terms that has a GCF of x .
 - Write a polynomial with two terms that has a GCF of $5y^2$.
 - Write a polynomial with three terms that has a GCF of $2a^2b^3$.

8. The formula for the perimeter of a rectangle is $P = 2l + 2w$.
- Write the formula in factored form.
 - If l represents 15 cm and w represents 9 cm, find the perimeter using both the original and the factored forms. What do you notice? Explain why this is so.
9. The formula for the surface area of a cylinder is $SA = 2\pi r^2 + 2\pi rh$.
- Write the formula in factored form.
 - If r represents 3 cm and h represents 8 cm, find the surface area using both the original and the factored forms. What do you notice? Explain why this is so.
10. A rectangle has area given by the expression $6x^2 + 9x$. The length and width can be found by factoring the expression. Find all possible expressions for the length and width.
11. Binomial factors can differ by a factor of -1 . An example is $7 - y$ and $y - 7$, since $7 - y$ can be rewritten as $-1(y - 7)$. Use this fact to factor each expression.
- $5x(7 - y) + 4(y - 7)$
 - $5y(x - 1) + 2(1 - x)$

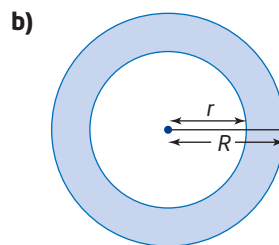
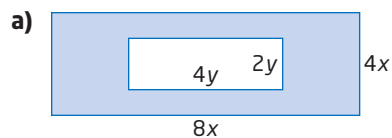
12. Chapter Problem

The base length of each square-based prism in the pedestal design is 3 cm less than that of the layer immediately below.



- Write an algebraic expression for the total of the top surface areas of the three prisms used to make the pedestal.
- Expand and simplify.
- Factor the resulting expression from part b).

13. Write an expression, in fully factored form, for each of the shaded regions.



Extend

14. Factor the quadratic relation $y = 2x^2 - 3x$ to find the x-intercepts.
15. Factor each polynomial using a fraction as one of the common factors. Explain how this can simplify the operations when the values of the variables are known.
- $\frac{1}{2}x^2 + \frac{3}{2}y^2$
 - $\frac{2}{3}a^3 - \frac{1}{3}ab$
 - $\frac{1}{6}k^4m^2 - \frac{1}{2}km^3 + \frac{1}{3}k^2m^2$
16. **Math Contest** If $3a + 8b = 12$, then what is the value of $15a + 40b$?
- 36
 - 48
 - 60
 - 84
 - 180
17. **Math Contest** Show that the sum of the squares of any five consecutive integers is divisible by 5.