# 6.2 Solve Quadratic Equations

The flight path of a toy rocket, a ball, or any projectile can be predicted using a quadratic model. This model can also be used to determine when and where a projectile will land. For example, the path of a stone thrown into a ravine is modelled by the quadratic relation $y = -x^2 + 5x + 84$, where $x$ is the distance, in metres, it travels horizontally and $y$ is the height, in metres, above the river at the bottom of the ravine. You can solve for $x$ using a variety of methods, including factoring a **quadratic equation**.

**quadratic equation**

- an equation in the form $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$

## Investigate

### How can factoring help you solve a quadratic equation?

1. If the product of two numbers is zero, what must be true about one or both of the numbers?
2. If $a \times b = 0$, what must be true about the value of $a$ or the value of $b$ or both?
3. How do steps 1 and 2 relate to solving the equation $(x - 3)(x + 5) = 0$?
4. **a)** Solve the equation $(x - 3)(x + 5) = 0$.
   **b)** Explain why your solutions cause the left side of the equation to equal zero.
5. Use your method to solve each equation.
   **a)** $(x + 2)(x + 9) = 0$ **b)** $(2x + 5)(3x - 4) = 0$
6. **a)** How is the equation $x^2 + 6x + 8 = 0$ different from those in steps 3 to 5? Describe the extra steps needed to solve the equation.
   **b)** Solve the equation.
7. **Reflect** Describe how you can use factoring to solve a quadratic equation.

## Example 1 Solve by Factoring

Solve for $x$. Check your answers by substitution.

**a)** $x^2 + 9x + 14 = 0$

**b)** $2x^2 + 5x = 0$

**c)** $6x^2 - x = 15$

### Solution

**a)** $x^2 + 9x + 14 = 0$

$(x + 7)(x + 2) = 0$ — Factor the left side.

$x + 7 = 0$ or $x + 2 = 0$ — One factor or the other must equal zero.

$x = -7$ or $x = -2$

Check.

For $x = -7$:

$\text{L.S.} = x^2 + 9x + 14$ $\qquad$ $\text{R.S.} = 0$

$= (-7)^2 + 9(-7) + 14$

$= 49 - 63 + 14$

$= 0$

$\text{L.S.} = \text{R.S.}$

For $x = -2$:

$\text{L.S.} = x^2 + 9x + 14$ $\qquad$ $\text{R.S.} = 0$

$= (-2)^2 + 9(-2) + 14$

$= 4 - 18 + 14$

$= 0$

$\text{L.S.} = \text{R.S.}$

The solutions, or **roots**, are $-7$ and $-2$.

**b)** $2x^2 + 5x = 0$

$x(2x + 5) = 0$ — Remove the common factor.

$x = 0$ or $2x + 5 = 0$ — One factor or the other must equal zero.

$2x = -5$

$x = -\frac{5}{2}$

The roots are 0 and $-\frac{5}{2}$. The check of these roots is left to the reader.

**c)** $6x^2 - x = 15$

$6x^2 - x - 15 = 0$ — Write in the form $ax^2 + bx + c = 0$.

$6x^2 - 10x + 9x - 15 = 0$ — To factor the left side, break up the middle term and factor by grouping.

$2x(3x - 5) + 3(3x - 5) = 0$

$(3x - 5)(2x + 3) = 0$

$3x - 5 = 0$ or $2x + 3 = 0$ — One factor or the other must equal zero.

$3x = 5$ or $2x = -3$

$x = \frac{5}{3}$ or $x = -\frac{3}{2}$

**Literacy Connections**

To solve an equation means to find the values of the variable that make the statement true. This is also called finding the roots of the equation.

**root (of an equation)**

- the value of the variable that makes an equation true
- the same as the solution of an equation

To factor $6x^2 - x - 15$, I must find two integers whose product is $6 \times (-15)$, or $-90$, and whose sum is $-1$. The integers $-10$ and 9 work.

Check.

For $x = \frac{5}{3}$:

$$\begin{aligned}\textbf{L.S.} &= 6x^2 - x \qquad \textbf{R.S.} = 15\\ &= 6\left(\frac{5}{3}\right)^2 - \frac{5}{3}\\ &= 6\left(\frac{25}{9}\right) - \frac{5}{3}\\ &= \frac{50}{3} - \frac{5}{3}\\ &= 15\end{aligned}$$

$\textbf{L.S.} = \textbf{R.S.}$

For $x = -\frac{3}{2}$:

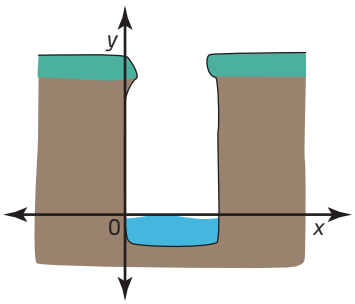
$$\begin{aligned}\textbf{L.S.} &= 6x^2 - x \qquad \textbf{R.S.} = 15\\ &= 6\left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)\\ &= 6\left(\frac{9}{4}\right) + \frac{3}{2}\\ &= \frac{27}{2} + \frac{3}{2}\\ &= 15\end{aligned}$$

$\textbf{L.S.} = \textbf{R.S.}$

The roots are $\frac{5}{3}$ and $-\frac{3}{2}$.

## Example 2 Path of a Stone

The path of a stone thrown into a ravine is modelled by the quadratic relation $y = -x^2 + 5x + 84$, where $x$ represents the distance, in metres, travelled horizontally and $y$ represents the height, in metres, above the surface of the river at the bottom of the ravine. How far does the stone travel horizontally before it hits the water?




### Solution

When the stone hits the water, its height is 0 m. So, let $y = 0$.

$$-x^2 + 5x + 84 = 0$$

$$\frac{-x^2}{-1} + \frac{5x}{-1} + \frac{84}{-1} = \frac{0}{-1}$$ Divide both sides of the equation by $-1$.

$$x^2 - 5x - 84 = 0$$

$$(x + 7)(x - 12) = 0$$ Factor the left side.

$$x + 7 = 0 \quad \text{or} \quad x - 12 = 0$$ One factor or the other must equal zero.

$$x = -7 \quad \text{or} \quad x = 12$$

Since $x$ represents a distance, it cannot be negative. So, reject the root $-7$.

Check $x = 12$.

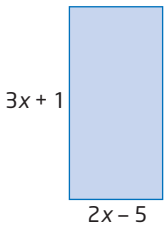
$$\begin{aligned}\textbf{L.S.} &= -x^2 + 5x + 84 \qquad \textbf{R.S.} = 0\\ &= -(12)^2 + 5(12) + 84\\ &= -144 + 60 + 84\\ &= 0\end{aligned}$$

$\textbf{L.S.} = \textbf{R.S.}$

The stone travelled 12 m horizontally before it hit the water.

## Example 3 Dimensions of a Rectangle

A rectangle has dimensions $3x + 1$ and $2x - 5$. Its area is 1150 cm$^2$. What are its dimensions?




### Solution

**Method 1: Use Pencil and Paper**

Substitute the area and expressions for the dimensions into the formula $A = l \times w$.

$$1150 = (3x + 1)(2x - 5)$$

$$1150 = 6x^2 - 13x - 5$$ Expand and simplify the right side.

$$6x^2 - 13x - 1155 = 0$$ Write in the form $ax^2 + bx + c = 0$.

$$6x^2 - 90x + 77x - 1155 = 0$$ Factor the left side.

$$6x(x - 15) + 77(x - 15) = 0$$

$$(x - 15)(6x + 77) = 0$$

$x - 15 = 0$ or $6x + 77 = 0$ One factor or the other must equal zero.

$x = 15$ or $x = -\frac{77}{6}$

Since the dimensions cannot be negative, reject the solution $-\frac{77}{6}$.

Check $x = 15$.

Find the dimensions of the rectangle.

$3x + 1$ $2x - 5$

$= 3(15) + 1$ $= 2(15) - 5$

$= 46$ $= 25$

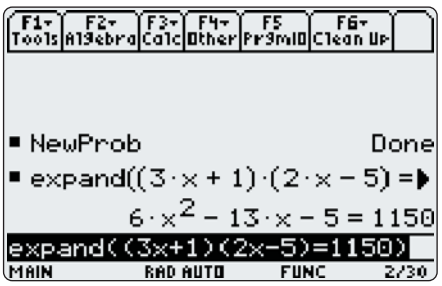
Check that these dimensions give an area of 1150 cm$^2$.

$46 \times 25 = 1150$

Therefore, the dimensions of the rectangle are 46 cm by 25 cm.

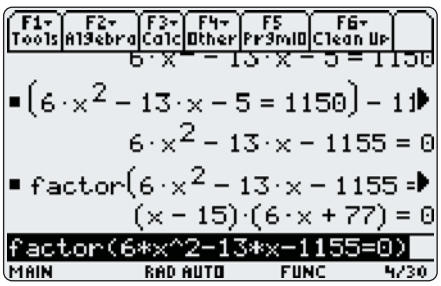
**Method 2: Use a Computer Algebra System (CAS)**

In the Home screen, use the **Expand** function with the area equation $(3x + 1)(2x - 5) = 1150$.



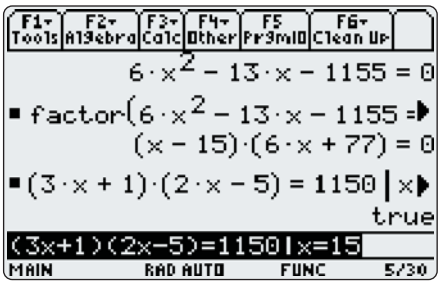
Express the equation in the form $ax^2 + bx + c = 0$.
Use the up cursor key to highlight the expanded form. Press ◆ ↑ for [COPY]. Cursor down to the command line. Press ◆ ESC for [PASTE]. Put brackets around the equation and subtract 1150. Press ENTER.

COPY and PASTE the new form of the equation into the command line. Cursor to the beginning of the equation. Press F2. Select **2:factor(**. At the end of the command line, press ) ENTER.



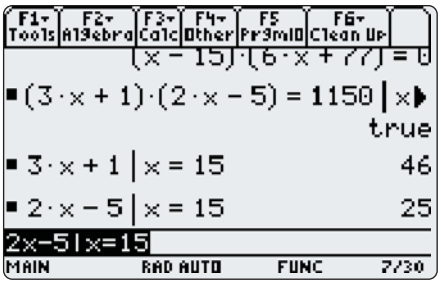

Note the two factors. The second factor results in a negative solution, and is rejected. Therefore, $x = 15$ is the solution.

To check the solution, type the original equation, followed by the *such that* symbol, | , and then $x = 15$. Press ENTER.




Note that the CAS returns a value of *true*.

Next, calculate the dimensions of the rectangle.
Type $3x + 1 \mid x = 15$ and press ENTER.
Type $2x - 5 \mid x = 15$ and press ENTER.




Check that these dimensions give an area of 1150 cm$^2$.
$46 \times 25 = 1150$
Therefore, the dimensions of the rectangle are 46 cm by 25 cm.

## Key Concepts

- If two factors have a product of zero, then one or both of the factors must equal zero.
- A quadratic equation has degree two and has a single unknown. For example, $x^2 + 2x - 3 = 0$ is a quadratic equation.
- Some quadratic equations can be solved by factoring.
- To solve a quadratic equation by factoring, first write the equation in the form $ax^2 + bx + c = 0$, and then factor the left side. Next, set each factor equal to zero, and solve for the unknown.

  For example,

$$\begin{aligned} x^2 + 2x &= 3 \\ x^2 + 2x - 3 &= 0 \\ (x + 3)(x - 1) &= 0 \\ x + 3 = 0 \quad &\text{or} \quad x - 1 = 0 \\ x = -3 \quad &\text{or} \quad x = 1 \end{aligned}$$

- The solutions to a quadratic equation are also known as the roots of the equation.

## Communicate Your Understanding

**C1** When you are factoring to solve a quadratic equation such as $x^2 + 2x - 3 = 0$, the right side needs to be equal to zero. Why?

**C2** Describe how you would solve each quadratic equation by factoring.

**a)** $3x^2 + 12x + 9 = 0$

**b)** $2x^2 - 11x = -15$

## Practise

*For help with questions 1 to 5, see Example 1.*

**1.** Solve.

**a)** $(x + 5)(x + 2) = 0$

**b)** $(x - 3)(x + 4) = 0$

**c)** $(x - 1)(x - 7) = 0$

**d)** $x(x + 9) = 0$

**e)** $(2x + 3)(x - 5) = 0$

**f)** $(2x - 1)(3x + 4) = 0$

**g)** $(3x - 5)(4x - 3) = 0$

**2.** Solve and check.

**a)** $x^2 + 8x + 12 = 0$

**b)** $h^2 + 9h + 18 = 0$

**c)** $m^2 + 3m = 0$

**d)** $w^2 - 18w + 56 = 0$

**e)** $x^2 - 2x = 0$

**f)** $c^2 - 17c + 30 = 0$

**g)** $n^2 + 9n - 22 = 0$

**h)** $y^2 - 11y = 0$

**3.** Solve.

**a)** $3x^2 + 28x + 9 = 0$

**b)** $4k^2 + 19k + 15 = 0$

**c)** $8y^2 - 22y + 15 = 0$

**d)** $16b^2 - 1 = 0$

**e)** $10m^2 + 30m = 0$

**f)** $4x^2 - 12x + 9 = 0$

**4.** Solve.

**a)** $x^2 + 5x = -4$

**b)** $8c + 15 = -c^2$

**c)** $k^2 = 13k - 12$

**d)** $b^2 + 1 = -2b$

**e)** $m^2 = 300 - 20m$

**f)** $y^2 = 7y$

**5.** Solve.

**a)** $2m^2 = -7m - 6$

**b)** $9x^2 = x + 8$

**c)** $4y^2 - 12y = -9$

**d)** $-5 = 2p - 16p^2$

**e)** $12m^2 = 10 - 37m$

**f)** $3w^2 + 22w = -7$

*For help with questions 6 and 7, see Example 2.*

**6.** Solve.

**a)** $-x^2 - 10x - 16 = 0$

**b)** $3t^2 + 24t + 45 = 0$

**c)** $6d^2 + 15d = -9$

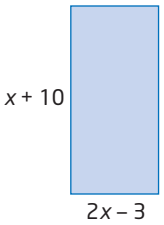
**d)** $-10g^2 + 32g = 6$

## Connect and Apply

**7.** A basketball is tossed from the top of a 3-m wall. The path of the basketball is defined by the relation $y = -x^2 + 2x + 3$, where $x$ represents the horizontal distance travelled, in metres, and $y$ represents the height, in metres, above the ground. How far has the basketball travelled horizontally when it lands on the ground?

*For help with question 8, see Example 3.*

**8.** A rectangle has dimensions $x + 10$ and $2x - 3$. Determine the value of $x$ that gives an area of 54 cm$^2$.




**9.** Write a quadratic equation in factored form for each situation.

**a)** The roots of the equation are 5 and 4.

**b)** The roots of the equation are $-2$ and 3.

**10.** **a)** Write a quadratic equation in the form $ax^2 + bx + c = 0$ with roots of 6 and $-7$.

**b)** What would happen to the roots if you multiplied both sides of the equation in part a) by 3? Explain.

**11.** Write a quadratic equation with roots of $\frac{2}{3}$ and $-\frac{4}{5}$ in the form $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are integers.

**12.** **a)** Create an example of a quadratic equation that can be factored and solved with integer solutions.

**b)** Create an example of a quadratic equation that can be factored and solved with non-integer solutions.

**13.** Create an example of a quadratic equation that cannot be solved by factoring. Explain why it cannot be factored.

**14.** The hypotenuse of a right triangle measures 29 cm. One leg is 1 cm shorter than the other. What are the lengths of the legs?

**15.** For the equation $3n^2 = 15n$, Chris suggested dividing both sides by $3n$, leaving $n = 5$. Are there any other values that satisfy this equation? What is wrong with Chris's method? What is a more appropriate method?

**16.** The sum of the first $n$ even natural numbers can be found using the formula $S = n(n + 1)$.

**a)** Verify the formula for $n = 1$ and $n = 2$.

**b)** What is the sum of the first five even natural numbers?

**c)** The sum of the first $n$ even natural numbers is 306. What is the value of $n$?

**17.** At the Mini Market, as the price of milk drops, sales increase. On an average day, a 4-L bag of milk costs \$3.90, and the store sells an average of 120 bags. For each \$0.10 reduction in price of a 4-L bag, sales increase by 20 bags per day. The price and value of sales can be modelled as follows, where $n$ is the number of \$0.10 price reductions.

Price, in dollars: $3.90 - 0.10n$
Number of bags: $120 + 20n$

The total revenue is the product of the price and the number of bags sold. Find how many price reductions will result in revenue of \$700.

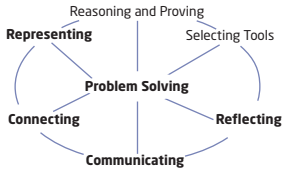
### Did You Know?

According to the Canadian Dairy Information Centre, in 2005 the per capita consumption of fluid milk in Canada was 84.00 L. Fluid milk includes 3.25%, 2%, 1%, skim, chocolate, and buttermilk.

For the same year, Finland had the greatest per capita retail milk consumption, 116.99 L, while China had the least, 2.55 L.

## Achievement Check

**18.** The length of the base of a rectangular prism is 2 m greater than the width, and the height of the prism is 15 m.




**a)** Write an algebraic expression for the volume of the rectangular prism. Express the relation in the form $y = a(x - h)^2 + k$.

**b)** If the volume of the prism is 2145 $m^3$, write an equation to model the situation.

**c)** Solve the equation by factoring. What are the dimensions of the base of the rectangular prism?

**d)** Make up a similar problem. Trade with a partner and solve each other's problem.

## Extend

**19.** The length of a rectangular plot of land is 7 m greater than its width. The diagonal is 8 m greater than the width of the plot of land. What are the dimensions of the plot of land?

**20.** Ralph is opening a BMX bike repair shop. His accountant models his profit, $P$, with the equation $P = 1125(t - 1)^2 - 4500$, where $t$ is the number of years of operation. During the first 5 years of operation, when is Ralph's shop predicted to make a loss? a profit?

**21. a)** Solve the equation $y^2 + 3y + 2 = 0$. How is the equation $y^2 + 3yx + 2x^2 = 0$ related?

**b)** Solve for $y$ in terms of $x$.

**i)** $y^2 + 3yx + 2x^2 = 0$

**ii)** $5y^2 - 6yx - 8x^2 = 0$

**iii)** $\frac{1}{9}y^2 - \frac{1}{3}yx + \frac{1}{4}x^2 = 0$