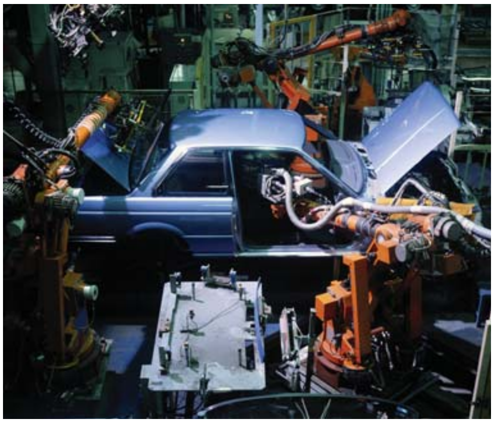
2.1 Midpoint of a Line Segment

Cartesian grid

- grid with perpendicular axes

Coordinates on a **Cartesian grid** are a simple and convenient way to specify a location. Such grids have many uses beyond graphs and maps. Machinists use coordinates to direct computer numerical control (CNC) machine tools that shape, drill, and weld parts. Similarly, coordinate systems are essential for programming industrial robots on automated assembly lines.



When you know the coordinates of a set of points, you can calculate the coordinates of related points, such as the **midpoint** between a pair of points.

midpoint

- point that divides a line segment into two equal line segments

Investigate

Tools

- grid paper

How can you determine the coordinates of a midpoint?

Method 1: Use Pencil and Paper

1. Use a Cartesian grid to plot the line segment defined by each pair of endpoints. Label the endpoints with their coordinates. What property do the line segments have in common?

a) A(−4, 2) and B(6, 2) **b)** C(−3, 0) and D(2, 0)

c) E(5, −2) and F(−4, −2)

2. Count squares or use a ruler to determine the coordinates of the midpoint of each segment. Label each midpoint with its coordinates. How are the coordinates of the midpoint of each line segment related to the coordinates of its endpoints?

3. Plot and label the line segment defined by each pair of endpoints. What property do the line segments have in common?

a) G(−4, 2) and H(−4, −6) **b)** J(−1, 7) and K(−1, −2)

c) L(5, −4) and N(5, 7)

4. Determine the coordinates of the midpoint of each line segment in step 3. Label each midpoint with its coordinates. How are the coordinates of the midpoint of each segment related to the coordinates of its endpoints?

5. Plot and label the line segment defined by each pair of endpoints.

a) P(1, 1) and Q(7, 5) **b)** R(−5, −4) and S(−1, 0)

c) T(−3, −4) and U(6, 1) **d)** V(−4, 6) and W(3, 4)

6. Determine the coordinates of the midpoint of each line segment in step 5. Describe how you calculated these coordinates.

7. **Reflect** How are the coordinates of the midpoint of a line segment related to the coordinates of the endpoints? Write an expression for the coordinates of the midpoint of a line segment that has endpoints at (x_1, y_1) and (x_2, y_2).

Method 2: Use *The Geometer's Sketchpad*®

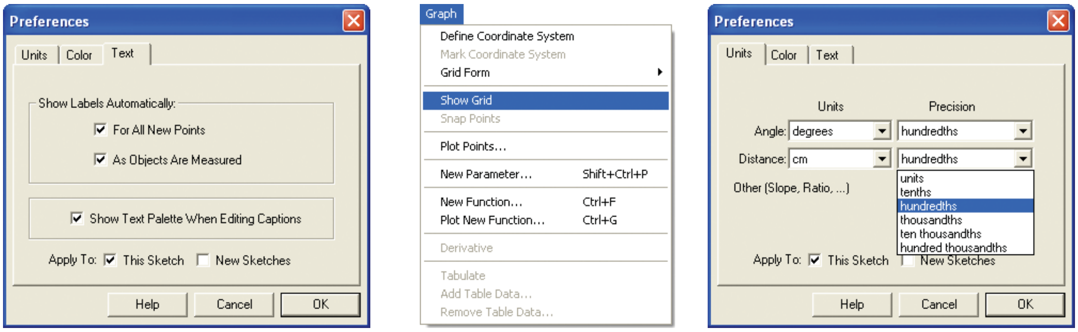
Tools

- computer with *The Geometer's Sketchpad*®

1. Turn on automatic labelling of points. From the **Edit** menu, choose **Preferences**. Click on the **Text** tab. Ensure that **For All New Points** is checked.

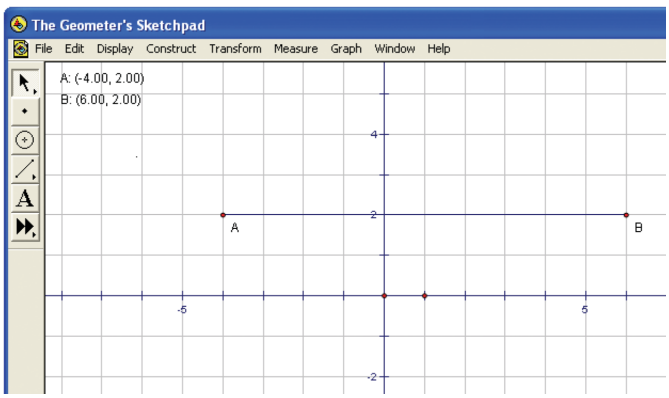
To display a Cartesian grid, open the **Graph** menu and choose **Show Grid**.

From the **Edit** menu, choose **Preferences;** for distance, choose **cm** units and **hundredths** precision.



2. From the **Graph** menu, choose **Plot Points**. Enter the coordinates for the point A(−4, 2). Use the tab key to move from the x-coordinate to the y-coordinate. Then, plot the point B(6, 2). To display the coordinates of points A and B, select the points and choose **Coordinates** from the **Measure** menu.

3. Construct the line segment AB by selecting points A and B and then choosing **Segment** from the **Construct** menu. Predict the coordinates of the midpoint of this line segment.



4. Select line segment AB and choose **Midpoint** from the **Construct** menu. Then, select the midpoint and choose **Coordinates** from the **Measure** menu. Check whether the coordinates of the midpoint match your prediction.

5. Construct a line segment with endpoints D(−3, 0) and E(2, 0). Predict the coordinates of the midpoint of DE. Then, use *The Geometer's Sketchpad*® to determine these coordinates. Predict and then determine the coordinates of the midpoint of a line segment with endpoints G(5, −2) and H(−4, −2).

6. **Reflect** What property do line segments AB, DE, and GH have in common? How are the coordinates of their midpoints related to the coordinates of their endpoints?

7. Start a new sketch. Predict and then determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.

   a) A(−4, 2) and B(−4, −6) **b)** D(−1, 7) and E(−1, −2)

   c) G(5, −4) and H(5, 7)

8. **Reflect** What property do line segments AB, DE, and GH have in common? How are the coordinates of their midpoints related to the coordinates of their endpoints?

9. Start a new sketch. Predict and then determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.

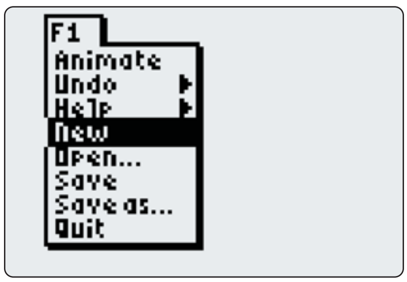
   a) A(1, 1) and B(7, 5) **b)** D(−5, −4) and E(−1, 0)

   c) G(−3,−4) and H(6, 1) **d)** J(−4, 6) and K(3, 4)

10. **Reflect** How are the coordinates of the midpoint of a line segment related to the coordinates of the endpoints? Write an expression for the coordinates of the midpoint of a line segment that has endpoints at (x_1, y_1) and (x_2, y_2).

Method 3: Use a Graphing Calculator

1. Press APPS, and choose **CabriJr**. Press ENTER when the title screen appears. If you need to clear the screen, press Y= to display the **F1** menu, and choose **New**.



Tools

- TI-83 Plus or TI-84 Plus graphing calculator

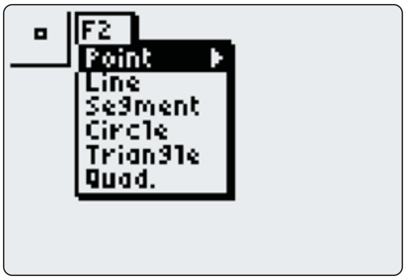
Technology Tip

The position where **CabriJr** appears on the **APPS** screen depends on what other applications have been installed.

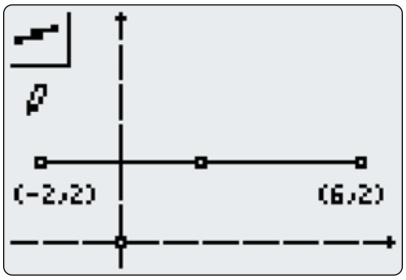
2. If the axes do not appear on the screen, press GRAPH to display the **F5** menu. Highlight **Hide/Show**, press ▸, and choose **Axes**. Press ENTER and then CLEAR. Position the axes so that you can graph the points (−2, 2) and (6, 2). Move the cursor close to the axes. When the axes start flashing, press ALPHA. Then, use the cursor keys to move the axes to the desired position and press ALPHA again.

Go to www.mcgrawhill.ca/links/principles10 and follow the links to install or upgrade Cabri® Jr.

3. Press WINDOW to display the **F2** menu. Move the cursor up to **Point** if it is not already highlighted and press ENTER. Press GRAPH to display the **F5** menu, choose **Coord.&Eq.**, and press ENTER twice to show the coordinates of the plotted point. You can use the cursor keys to move the coordinates label. Press ENTER and then CLEAR. If the coordinates are not exactly (−2, 2), move the cursor to the point. When the point flashes, press ALPHA. Then, use the cursor keys to reposition the point and press ALPHA again.



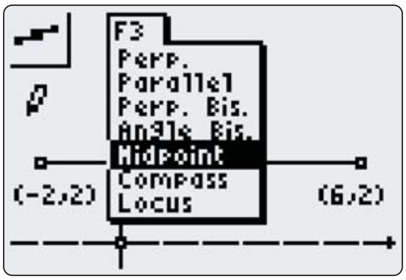
4. Plot and label a point at (6, 2). Press WINDOW to display the **F2** menu. Choose **Segment**, move the cursor to one of the points, and press ENTER. Move the cursor to the other point, press ENTER again, and press CLEAR. Predict the coordinates of the midpoint of the line segment joining the two points.



Technology Tip

Small gaps mark units along the axes.

5. Press ZOOM to display the **F3** menu. Choose **Midpoint** and press ENTER. Move the cursor to the line segment. When the line segment starts flashing, press ENTER again. Now, move the cursor to the midpoint of the segment. Press GRAPH to display the **F5** menu. Choose **Coord.&Eq.**, press ENTER twice, and press CLEAR. Check whether the coordinates of the midpoint match your prediction.



6. Construct a line segment with endpoints (−3, 0) and (2, 0). Predict the coordinates of the midpoint of this line segment. Then, use Cabri® Jr. to determine these coordinates. Predict and then determine the coordinates of the midpoint of a line segment with endpoints (5, −2) and (−2, −2).

7. **Reflect** What property do the three line segments on the screen have in common? How are the coordinates of their midpoints related to the coordinates of their endpoints?

Technology Tip
You can also use the cursor keys to switch between options in a dialogue box.

8. To clear the screen, press Y= and choose **New**. Press 2nd to highlight **NO**, and then press ENTER.

9. Predict and then determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.
   a) (−4, 4) and (−4, −2) **b)** (−1, 3) and (−1, −1)
   c) (5, −2) and (5, 3)

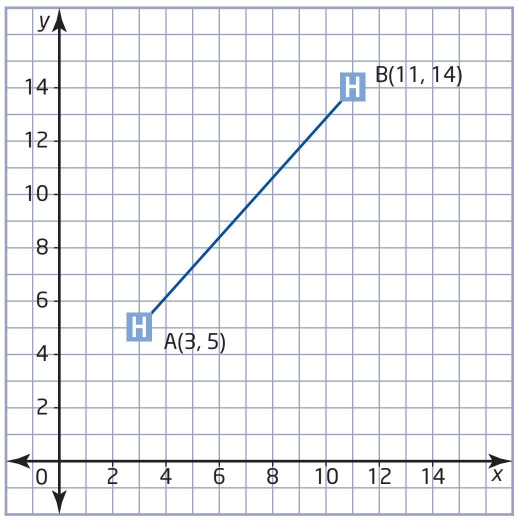
10. **Reflect** What property do the three line segments in step 9 have in common? How are the coordinates of their midpoints related to the coordinates of their endpoints?

11. Start a new graph. Predict and then determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.
    a) (−2, 2) and (4, 4) **b)** (1, 0) and (5, 3)
    c) (−2, −1) and (4, −2)

12. **Reflect** How are the coordinates of the midpoint of a line segment related to the coordinates of the endpoints? Write an expression for the coordinates of the midpoint of a line segment that has endpoints at (x_1, y_1) and (x_2, y_2).

Example 1 Find a Midpoint

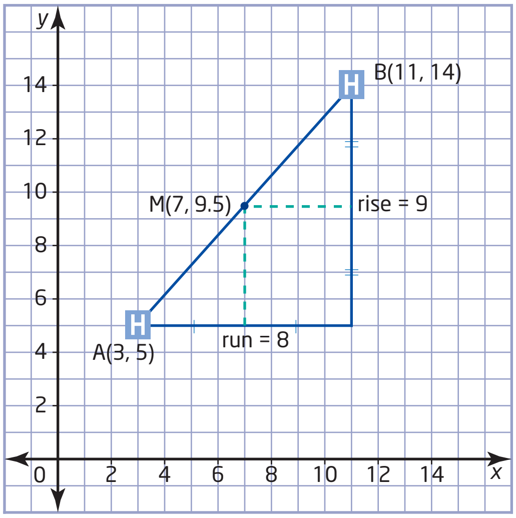
A city has two hospitals, shown on the city map at coordinates A(3, 5) and B(11, 14). The city wants to build a new ambulance station halfway between the two hospitals. Determine the coordinates of this location.



Solution

Method 1: Calculate the Rise and Run

The location of the new ambulance station is the midpoint of the line segment AB. The run between point A and the midpoint is half the run of AB. Similarly, the rise between point A and the midpoint is half the rise of AB.



$$\begin{aligned} \text{run} &= x_2 - x_1 \\ &= 11 - 3 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \text{rise} &= y_2 - y_1 \\ &= 14 - 5 \\ &= 9 \end{aligned}$$

I could use a similar method to divide a line segment into three or more equal parts.

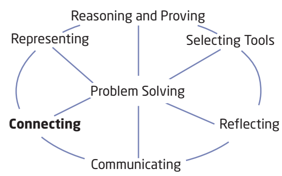
Therefore, the coordinates of the midpoint are

$$\begin{aligned} (x, y) &= \left(x_1 + \frac{\text{run}}{2}, y_1 + \frac{\text{rise}}{2}\right) \\ &= \left(3 + \frac{8}{2}, 5 + \frac{9}{2}\right) \\ &= (3 + 4, 5 + 4.5) \\ &= (7, 9.5) \end{aligned}$$

I can also subtract half the run and half the rise from the coordinates of the second point.

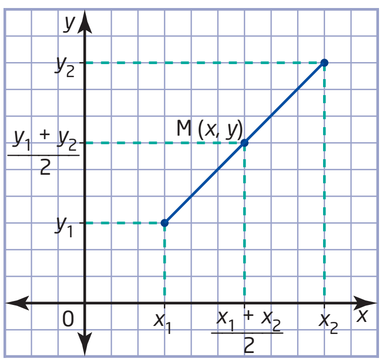
Method 2: Use a Formula

The x-coordinate of the midpoint is equal to the x-coordinate of point A plus half the difference between the x-coordinate of point B and the x-coordinate of point A. So, the x-coordinate of the midpoint is the mean of the x-coordinates of the endpoints of AB. Similarly, the y-coordinate of the midpoint is the mean of the y-coordinates of the endpoints.



Therefore, the coordinates of the midpoint are

$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{3 + 11}{2}, \frac{5 + 14}{2}\right) \\ &= \left(\frac{14}{2}, \frac{19}{2}\right) \\ &= (7, 9.5) \end{aligned}$$



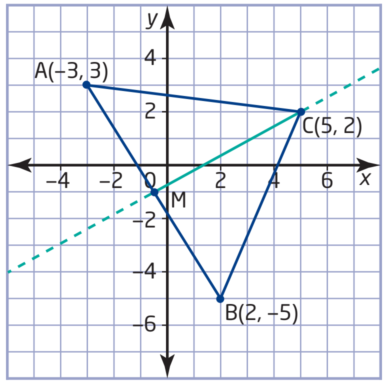
The coordinates of the new ambulance station are (7, 9.5).

Example 2 Median of a Triangle

median

- line segment joining a vertex of a triangle to the midpoint of the opposite side

Determine an equation for the **median** from vertex C for the triangle with vertices C(5, 2), A(−3, 3), and B(2, −5).



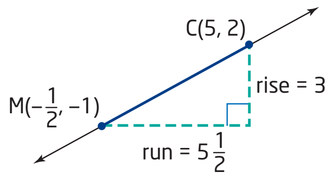
Solution

The median from C goes to the midpoint, M, of the opposite side, AB.

Use the formula from Example 1 to determine the coordinates of this midpoint.

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{-3 + 2}{2}, \frac{3 + (-5)}{2}\right)$$
$$= \left(\frac{-1}{2}, \frac{-2}{2}\right)$$
$$= \left(-\frac{1}{2}, -1\right)$$

Now, find the slope of CM.



$$\text{Slope, } m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - (-1)}{5 - \left(-\frac{1}{2}\right)}$$
$$= \frac{3}{\frac{11}{2}}$$
$$= 3 \times \frac{2}{11}$$
$$= \frac{6}{11}$$

If I extend the median on the graph, I can verify that its slope is $\frac{6}{11}$ by checking that the rise is 6 over a run of 11.

Since the point C(5, 2) is on the median, $y = 2$ when $x = 5$. Use these coordinates and the slope to solve for the y-intercept, b.

$$y = mx + b$$
$$2 = \frac{6}{11}(5) + b$$
$$2 = \frac{30}{11} + b$$
$$\frac{22}{11} = \frac{30}{11} + b$$
$$\frac{22}{11} - \frac{30}{11} = b$$
$$\frac{-8}{11} = b$$

I could use the coordinates of the midpoint, M, but the calculation with point A is easier since it has integer coordinates. I could use the coordinates of M to verify my answer.

The median intersects the y-axis at $y = -\frac{8}{11}$.

The y-intercept of the median is $-\frac{8}{11}$.

Therefore, an equation for the median from vertex C is $y = \frac{6}{11}x - \frac{8}{11}$.

Example 3 Equation of a Right Bisector

Two schools are located at the points P(−1, 4) and Q(7, −2) on a town map. The school board is planning a new sports complex to be used by both schools. The board wants to find a location **equidistant** from the two schools. Use an equation to represent the possible locations for the sports complex.

equidistant

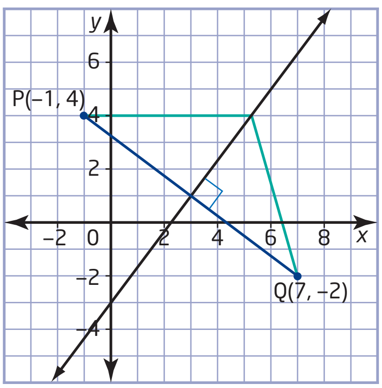
- equally distant

Solution

From the diagram, you can see that a point can be the same distance from both schools without being directly between them. In fact, any point on the **right bisector** of a line segment is equidistant from the endpoints of the segment. The possible locations for the athletic complex lie on the right bisector of PQ.

right bisector

- the line that passes through the midpoint of a line segment and intersects it at a 90° angle



Literacy Connections

A *right bisector* is sometimes called a *perpendicular bisector.*

To determine an equation for the right bisector, find the slope of the bisector and the coordinates of the midpoint of PQ. First, determine the slope of PQ.

$$\begin{aligned} m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - (-2)}{-1 - 7} \\ &= \frac{6}{-8} \\ &= -\frac{3}{4} \end{aligned}$$

Perpendicular lines have slopes that are the negative reciprocals of each other. So, the slope of any line perpendicular to PQ is

$$m_{\perp} = \frac{4}{3}$$

To find the negative reciprocal of a fraction, invert the fraction and use the opposite sign.

The right bisector passes through the midpoint of PQ. Use the midpoint formula to find the coordinates of the midpoint.

$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-1 + 7}{2}, \frac{4 + (-2)}{2}\right) \\ &= \left(\frac{6}{2}, \frac{2}{2}\right) \\ &= (3, 1) \end{aligned}$$

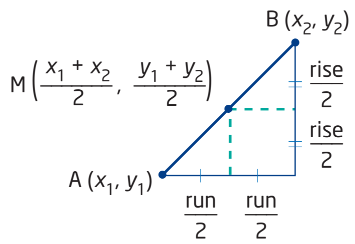
Now, use the coordinates of the midpoint with the slope to solve for the y-intercept of the right bisector.

$$\begin{aligned} y &= mx + b \\ 1 &= \frac{4}{3}(3) + b \\ 1 &= 4 + b \\ 1 - 4 &= b \\ -3 &= b \end{aligned}$$

I can use the graph to check that this value for the y-intercept is reasonable.

An equation for the right bisector of PQ is $y = \frac{4}{3}x - 3$. This equation represents the possible locations for the sports complex.

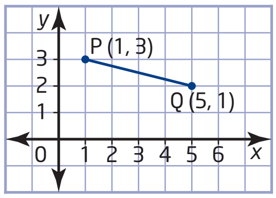
Key Concepts



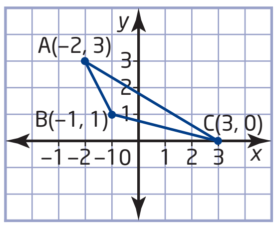
- The midpoint of a line segment can be found by adding half of the run and half of the rise to the coordinates of the first endpoint.
- Each coordinate of the midpoint of a line segment is the mean of the corresponding coordinates of the endpoints.
- The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- To find an equation for the median of a triangle, first find the coordinates of the midpoint of the side opposite the vertex. Use the coordinates of the midpoint and the vertex to calculate the slope of the median. Then, substitute the slope and the coordinates of either point into $y = mx + b$ to solve for the median's y-intercept.
- To find an equation for the right bisector of a line segment, first find the slope and midpoint of the segment. Use the line segment's slope to calculate the slope of a perpendicular line. Then, substitute this slope and the coordinates of the midpoint into $y = mx + b$ to solve for the right bisector's y-intercept.

Communicate Your Understanding

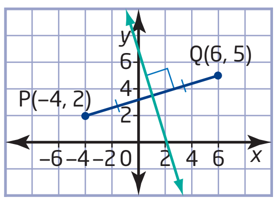
C1 Describe two methods for finding the midpoint of this line segment.



C2 Describe how to determine an equation for the median from vertex A of △ABC.



C3 Describe how to determine an equation for the right bisector of line segment PQ.

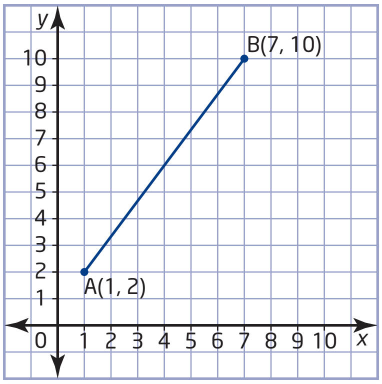


Practise

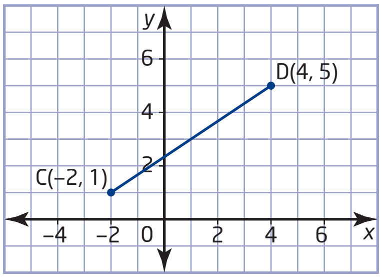
For help with questions 1 to 3, see Example 1.

1. Determine the coordinates of the midpoint of each line segment.

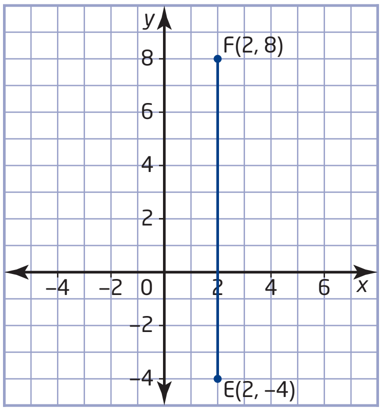
a)



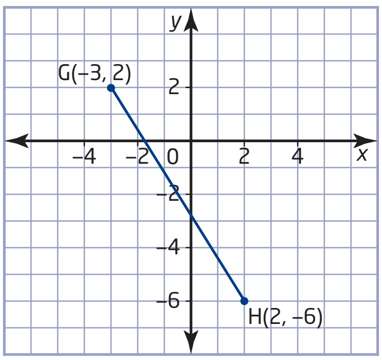
b)



c)



d)



2. Determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.

a) J(5, 7) and K(3, 9)

b) L(−1, 0) and M(1, −6)

c) N(−2, −4) and P(−2, 8)

d) Q(−3, −3) and R(−1, −7)

3. Determine the coordinates of the midpoint of the line segment defined by each pair of endpoints.

a) J(0.2, 1.5) and K(3.6, 0.2)

b) N(−1.4, −3.2) and P(0.6, −5.3)

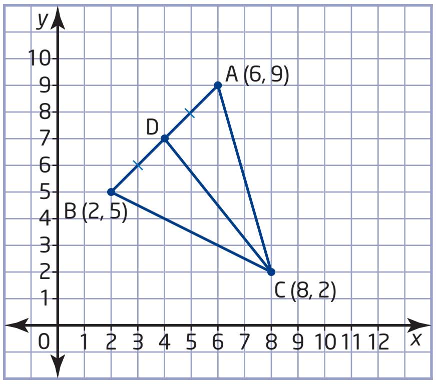
c) $L\left(\frac{1}{2}, \frac{5}{2}\right)$ and $M\left(\frac{3}{2}, -\frac{5}{2}\right)$

d) $Q\left(-\frac{3}{8}, \frac{1}{8}\right)$ and $R\left(2, -\frac{7}{8}\right)$

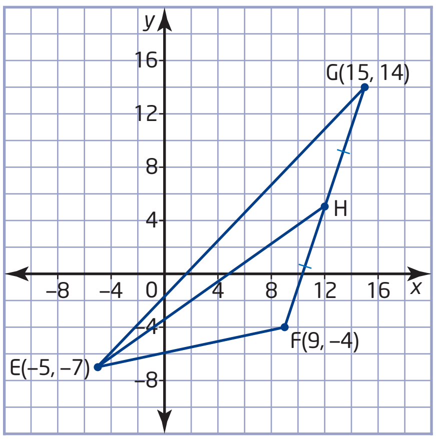
For help with question 4, see Example 2.

4. Find the slope of each median shown.

a)



b)



Connect and Apply

5. A charity is organizing a fundraising run along a straight section of highway. On the grid of a roadmap, the starting point is at (23.6, 38.0) and the finish line is at (79.4, 43.8). The charity wants to set up a checkpoint table with water for the runners at the halfway point. Find the coordinates of this checkpoint.

6. The endpoints of the diameter of a circle are P(−7, −4) and Q(−1, 10). Find the coordinates of the centre of this circle.

7. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 6. Describe the method you used.

8. The vertices of △ABC are A(4, 4), B(−6, 2), and C(2, 0). Find an equation in slope y-intercept form for the median from vertex A.

9. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 8. Describe the method you used.

Technology Tip

You can use geometry software to display an equation for a line:

- With *The Geometer's Sketchpad*®, choose **Equation** from the **Measure** menu.
- With Cabri® Jr., choose **Coord.&Eq.** from the **F5** menu.

10. For the triangle with vertices P(−2, 0), Q(4, 6), and R(5, −3), find an equation for the median from

a) vertex P **b)** vertex Q

11. **Use Technology** Use geometry software to check your answer to question 10. Describe your method.

12. Write an expression for the coordinates of the midpoint of the line segment with endpoints P(a, b) and Q($3a$, $2b$). Explain your reasoning.

13. A line segment with one end at C(6, 5) has midpoint M(4, 2).

a) Determine the coordinates of the other endpoint, D.

b) Explain your solution.

c) Describe a method you could use to check your answer to part a).

14. One endpoint of a diameter of a circle centred on the origin is (−3, 4). Find the coordinates of the other endpoint of this diameter.

15. One radius of a circle has endpoints D(2, 4) and E(−1, 2).

a) Find a possible endpoint for the diameter that contains this radius.

b) Explain why there are two possible answers in part a).

16. Determine an equation for the right bisector of the line segment with endpoints P(−5, −2) and Q(3, 6).

17. A telecommunications company wants to build a relay tower that is the same distance from two adjacent towns. On a local map, the towns have coordinates (2, 6) and (10, 0).

a) Explain how you could use a right bisector to find possible locations for the tower.

b) Find an equation for this bisector.

18. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 17. Describe the method you used.

19. **a)** Draw △ABC with vertices A(−2, 0), B(8, 8), and C(4, −2).

b) Draw the median from vertex A. Then, find an equation in slope y-intercept form for this median.

c) Draw the right bisector of BC. Then, find an equation for this right bisector.

d) Use your drawing to check your answers for parts b) and c).

20. **a)** Draw △PQR with vertices P(0, 0), Q(16, 0), and R(0, 16).

b) Construct the midpoints of PQ, QR, and PR, and label them S, T, and U, respectively.

c) Join the midpoints to form △STU. The length of a line segment joining the midpoints of two sides of a triangle is half the length of the third side. Use this property to show that △STU is congruent to all three of the other triangles inside △PQR.

d) Compare the area of △STU to the area of △PQR.

e) Shade △STU. Construct and label the midpoint of each side of the three other triangles inside △PQR. Join the midpoints to create a set of even smaller triangles.

f) Compare the area of one of these triangles to the area of △STU and to the area of △PQR.

21. **Chapter Problem** Question 20 uses a procedure developed by the Polish mathematician Waclaw Sierpinski in 1915. This procedure produces a fractal known as Sierpinski's triangle or Sierpinski's gasket.

a) Use a library or the Internet to learn more about Sierpinski's triangle.

b) Describe the procedure for producing this fractal. Does the procedure work with any shape of triangle? Explain.

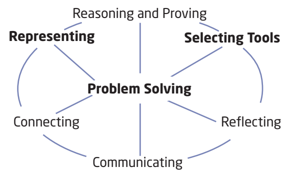
c) Sketch the first four stages in shading a triangle using Sierpinski's method.

d) Explain why Sierpinski's triangle is a fractal.

Go to www.mcgrawhill.ca/links/principles10 and follow the links to learn more about Sierpinski's triangle.

Extend

22. A is the midpoint of BC, D is the midpoint of AC, and E is the midpoint of AD. ED is 2 units in length. What is the length of BC?



23. A line segment has endpoints A(2, 1) and B(11, 19).

a) Find the coordinates of the two points that divide the segment into three equal parts.

b) Describe the method that you used in part a).

24. In △ABC, P(0, 2) is the midpoint of side AB, Q(2, 4) is the midpoint of BC, and R(1, 0) is the midpoint of AC.

a) Find the coordinates of A, B, and C. (Hint: Use the properties of a line segment joining the midpoints of two sides of a triangle.)

b) Use the midpoint formula to check the coordinates you calculated in part a).

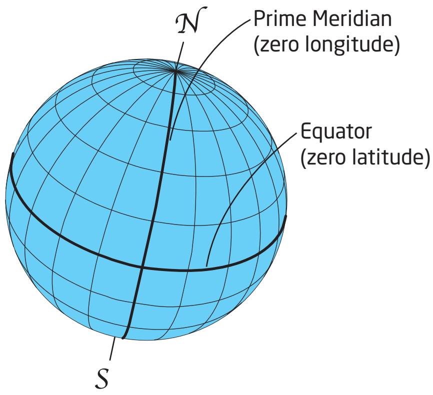
25. In three dimensions, the location of a point can be represented by the ordered triple (x, y, z).

a) Find the coordinates of the midpoint of the line segment with endpoints A(2, 3, 1) and B(6, 7, 5).

b) Write an expression for the coordinates of the midpoint of the line segment with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2).

26. Suppose that the relay tower in question 17 is to serve three towns instead of two. Describe how you could find a location that is equidistant from all three towns. Can there be more than one such location? Explain, using a diagram to support your answer.

27. Geographers and navigators use a spherical coordinate system with lines of latitude that are parallel to the equator and lines of longitude that are perpendicular to the lines of latitude and meet at Earth's poles.



a) Explain why the formula for the midpoint of a line segment will not always give accurate results with the longitude-latitude coordinate system.

b) Search the Internet to find a Web site that calculates the distance between two points from their latitudes and longitudes. Use the site's calculator to show that the point at the mean of the latitudes and longitudes of two points is not actually equidistant from the two points.

> Go to www.mcgrawhill.ca/links/principles10 and follow the links to experiment with distance calculators.

28. Decide whether each statement is always true, sometimes true, or never true. Justify your answers.

a) Two line segments with the same midpoint have the same length.

b) Two parallel line segments have the same midpoint.

c) The midpoint of a line segment is the only point that divides it into two equal parts.

d) A point equidistant from the endpoints of a line segment is the midpoint of the line segment.

29. The endpoints of line segment PQ are P(3, –4) and Q(11, c). The midpoint of PQ is M(d, 3). Find the values of c and d.

30. **Math Contest** The number of arrangements of five letters from the word *magnetic* that end with a vowel is

A 360 **B** 420 **C** 840 **D** 2520 **E** 7560

31. **Math Contest** If $2^5 + 2^5 = 2^x$, the value of x is

A 3 **B** 5 **C** 6 **D** 8 **E** 10

Making Connections

The branch of mathematics known as analytic geometry started when French mathematician René Descartes (1596–1650) invented a system of numerical coordinates for describing locations on a rectangular grid. The Cartesian grid let mathematicians use algebra to analyse geometric shapes. Before Descartes, all geometric properties were proved using logical reasoning based on the work of the Greek mathematician Euclid (around 300 B.C.E.) and his followers. Analytic geometry is the basis of applications such as computer-aided design (CAD) and computer-generated imaging (CGI). These applications are widely used for engineering, architecture, medical tests, cartoon animation, and special effects in movies.