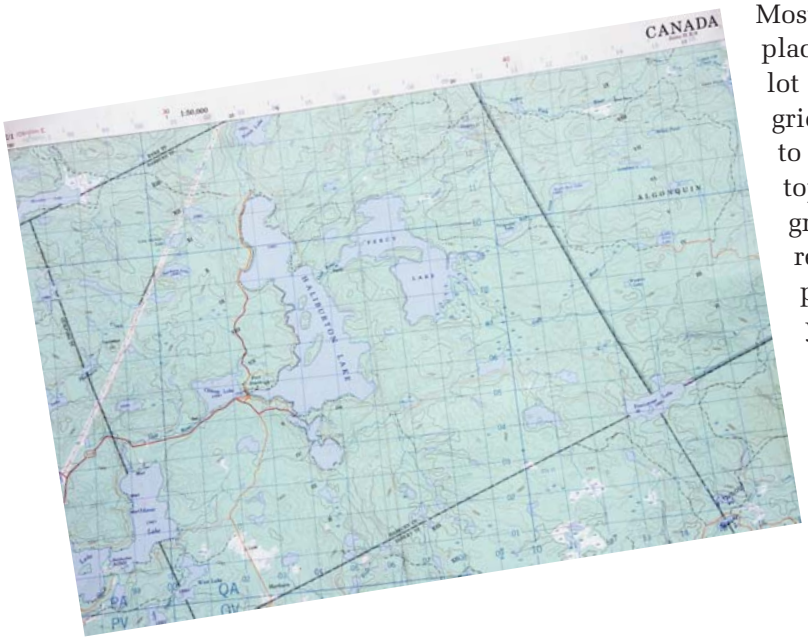
2.2 Length of a Line Segment

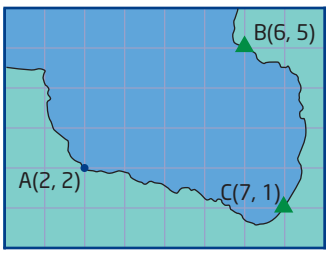


Most maps have a grid for locating places on the map. Maps that show a lot of detail usually have Cartesian grids with a scale that make it easy to estimate distances. For example, topographic maps commonly have grids where each side of a square represents 1 km. If you are planning a hiking or canoe trip, you can easily count squares to get a rough estimate of the length of possible routes. You can also use coordinates to calculate the exact distance between two points.

Investigate

How can you use coordinates to calculate distances?

Jan and Tara are planning a canoe trip in Georgian Bay. The dock where they will launch the canoe is at point A(2, 2) on their map. The campsite where they will stay the first night is at B(6, 5), and the campsite for the second night is at C(7, 1).



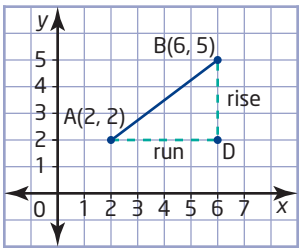
Tools

- grid paper

Method 1: Use Pencil and Paper

1. Plot points A and B on grid paper. Draw a line segment joining points A and B. What does this line segment represent?

2. Draw a right triangle below line segment AB to show its run and rise. How are the run and rise of AB related to the coordinates of its endpoints?



3. Find the coordinates of the vertex of the right angle in the run-rise triangle. Label this vertex D. How are the coordinates of D related to the coordinates of points A and B?

4. Find the lengths of sides AD and BD. Then, use the Pythagorean theorem to determine the length of AB.

5. Each side of a grid square represents an actual distance of 4.0 km. How far will Jan and Tara have to paddle on the first day of their trip?

6. **Reflect** How is the length of line segment AB related to its run and rise? How is this length related to the coordinates of points A and B?

7. Calculate the distance from the campsite at B to the campsite at C. Do you have to change your method for calculating the length if the run or rise is negative? Explain.

8. Calculate the distance from the campsite at C back to the dock at A.

9. **Reflect** How is the length, d, of the line segment with endpoints (x_1, y_1) and (x_2, y_2) related to the run and rise of the segment? Write an equation showing how d is related to the coordinates of the endpoints.

Tools

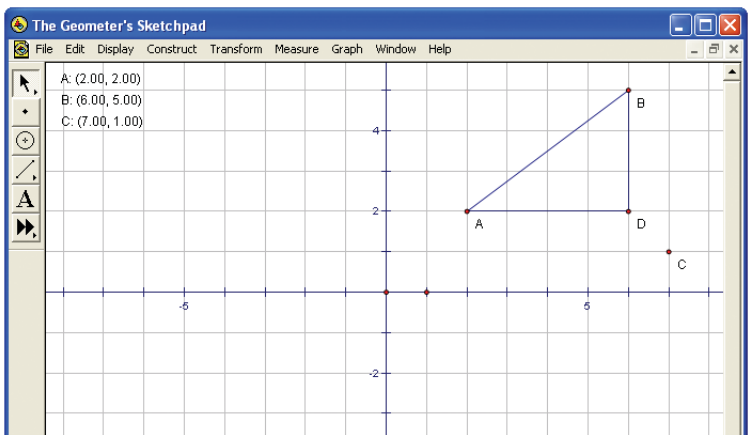
- computer with *The Geometer's Sketchpad®*

Method 2: Use *The Geometer's Sketchpad®*

1. From the **Graph** menu, choose **Show Grid**. From the **Edit** menu, choose **Preferences**; for distance, choose **cm** units and **hundredths** precision. Check that automatic labelling of points is turned on.

2. From the **Graph** menu, choose **Plot Points**. Then, enter the coordinates for A(2, 2), B(6, 5), and C(7, 1). Display the coordinates by opening the **Measure** menu and choosing **Coordinates**. Construct line segment AB.

3. Construct a right triangle below line segment AB to show its run and rise. How are the run and rise of AB related to the coordinates of its endpoints?



4. Display the coordinates of the vertex, D, of the right angle in the run-rise triangle. How are they related to the coordinates of points A and B?

5. Calculate the lengths of sides AD and BD. Then, use the Pythagorean theorem to determine the length of AB. Check your calculations by selecting the segments and choosing **Length** from the **Measure** menu.

6. Each side of a grid square represents an actual distance of 4.0 km. How far will Jan and Tara have to paddle on the first day of their trip?

7. **Reflect** How is the length of line segment AB related to its run and rise? How is this length related to the coordinates of points A and B?

8. Use the Pythagorean theorem to calculate the distance from the campsite at B to the campsite at C. Use the **Measure** menu to check your answer. Do you have to change your method for calculating the length if the run or rise is negative? Explain.

9. Calculate the distance from the campsite at C back to the dock at A. Use the **Measure** menu to check your answer.

10. **Reflect** How is the length, d, of the line segment with endpoints (x_1, y_1) and (x_2, y_2) related to the run and rise of the segment? Write an equation showing how d is related to the coordinates of the endpoints.

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

Method 3: Use a Graphing Calculator

1. Press APPS and choose **CabriJr**. Press ENTER when the title screen appears. If you need to clear the screen, press Y= to display the **F1** menu and choose **New**.

2. Press GRAPH to display the **F5** menu. Choose **Hide/Show**, press ▶, and choose **Axes**. Press ENTER and then CLEAR. Move the cursor close to the axes. When the axes start flashing, press ALPHA. Then, use the cursor keys to move the axes to the lower left corner of the screen. Press ALPHA again.

3. Press WINDOW to display the **F2** menu. Choose **Point**. Use the cursor keys to move the pen cursor to (2, 2) and press ENTER. Press GRAPH to display the **F5** menu, choose **Coord.&Eq.**, and press ENTER twice to show the coordinates of the plotted point. You can use the cursor keys to move the coordinates label. Press ENTER and then CLEAR. If the coordinates are not exactly (2, 2), move the cursor to the point A. When the point flashes, press ALPHA. Then, use the cursor keys to reposition the point and press ALPHA again.

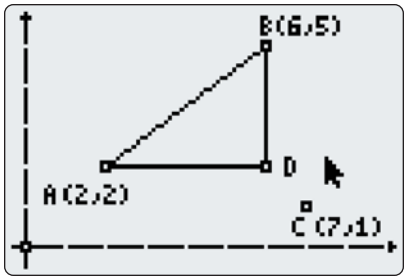
4. To add a letter label to a point, choose **Alph-Num** from the **F5** menu. Move the cursor near the point. Press ENTER and then ALPHA followed by the key for the letter. Then, press ENTER to lock the label in place. To move the label, press CLEAR and move the cursor toward the letter until it starts flashing. Then, press ALPHA, use the cursor keys to move the label to the desired location, and press ENTER.

Technology Tip

When moving a label or geometric shape, you can press either ENTER or ALPHA to set the position.

5. Plot and label points B(6, 5) and C(7,1). Press WINDOW to display the **F2** menu. Choose **Segment**, move the cursor to A(2, 2), and press ENTER. Move the cursor to B(6, 5), press ENTER again, and then press CLEAR.

6. Use the **F2** menu to construct a point D and two line segments forming a right triangle that shows the run and rise of line segment AB. How are the run and rise of segment AB related to the coordinates of its endpoints? How are the coordinates of point D, the vertex of the right angle, related to the coordinates of points A and B?



7. Find the lengths of sides AD and BD. Then, use the Pythagorean theorem to calculate the length of AB. To check this calculation, choose **Measure** from the **F5** menu and then choose **D.&Length**. Move the cursor toward segment AB until it flashes, and press ENTER. Use the cursor keys to position the measurement, if necessary; then, press ENTER again.

8. Each side of a grid square represents an actual distance of 4.0 km. How far will Jan and Tara have to paddle on the first day of their trip?

9. **Reflect** How is the length of line segment AB related to its run and rise? How is this length related to the coordinates of points A and B?

10. Use the Pythagorean theorem to calculate the distance from the campsite at B to the campsite at C. Measure the length as described above to check your answer. Do you have to change your method for calculating the length if the run or rise is negative? Explain.

11. Calculate the distance from the campsite at C back to the dock at A. Measure the length to check your answer.

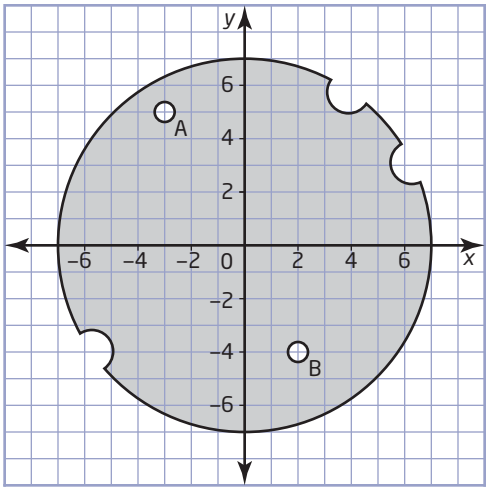
12. **Reflect** How is the length, d, of the line segment with endpoints (x_1, y_1) and (x_2, y_2) related to the run and rise of the segment? Write an equation showing how d is related to the coordinates of the endpoints.

Example 1 Calculate a Length

Literacy Connections

A cam is a rotating mechanical part shaped to guide the motion of another part of a machine. Usually, the controlled part connects to the side of the cam or runs against its edge.

To make round parts, programmable machine tools often use a coordinate system with the origin at the centre of the part. How far apart are the centres of the mounting holes A and B in this cam? The coordinates are in centimetres. Round your answer to the nearest tenth.

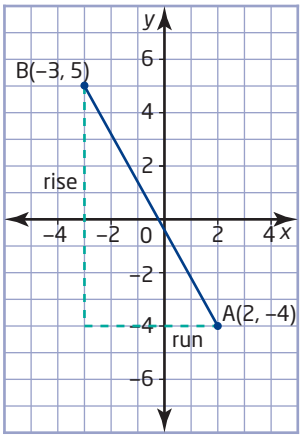


Solution

Applying the Pythagorean theorem gives $AB^2 = \text{run}^2 + \text{rise}^2$. The run of AB is equal to the difference between the x-coordinate of B and the x-coordinate of A. Similarly, the rise of AB is equal to the difference between the corresponding y-coordinates. Therefore,

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-3 - 2)^2 + [5 - (-4)]^2} \\ &= \sqrt{(-5)^2 + 9^2} \\ &= \sqrt{25 + 81} \\ &= \sqrt{106} \\ &\doteq 10.3 \end{aligned}$$

The centres of the mounting holes are about 10.3 cm apart.



Example 2 Compare Distances

An air ambulance service uses a grid system to help estimate flying times and fuel requirements. Coordinates on this grid are distances in kilometres east and north of a reference point on the lower left corner of a map of northern Ontario. A helicopter ambulance picks up a patient at point P(96, 197). The nearest hospitals that can provide the treatment the patient needs are in Timmins at T(200, 296) and Sudbury at S(232, 80).

a) To which hospital should the helicopter take the patient?

b) List any assumptions you made for your answer.

Solution

a) First, find the distance to each hospital.

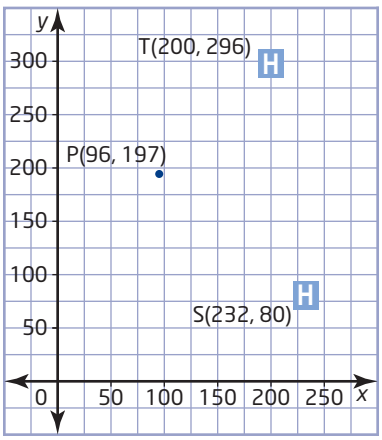
For the Timmins hospital:

$$\begin{aligned} PT &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(200 - 96)^2 + (296 - 197)^2} \\ &= \sqrt{104^2 + 99^2} \\ &= \sqrt{20\,617} \\ &\doteq 144 \end{aligned}$$

For the Sudbury hospital:

$$\begin{aligned} PS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(232 - 96)^2 + (80 - 197)^2} \\ &= \sqrt{136^2 + (-117)^2} \\ &= \sqrt{32\,185} \\ &\doteq 179 \end{aligned}$$

The helicopter should go to the Timmins hospital because it is closer to the pick-up point.



b) The decision to go to the closer hospital assumes that the helicopter can travel in a straight line to either hospital. The decision also assumes that weather will not affect the flying times or prevent a landing at the closer hospital.

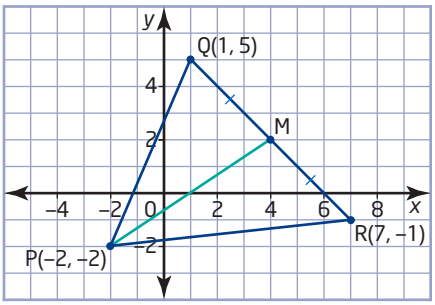
Example 3 Find the Length of a Median

Find the length of the median from P for a triangle with vertices P(−2, −2), Q(7, −1), and R(1, 5).

Solution

The median is the line segment that joins P to the midpoint, M, of QR. To find the coordinates of M, substitute the coordinates of Q and R into the midpoint formula.

$$\begin{aligned}(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\\ &= \left(\frac{7 + 1}{2}, \frac{-1 + 5}{2}\right)\\ &= \left(\frac{8}{2}, \frac{4}{2}\right)\\ &= (4, 2)\end{aligned}$$



Now, substitute the coordinates of P(−2, −2) and M(4, 2) into the length formula.

$$\begin{aligned}PM &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= \sqrt{[4 - (-2)]^2 + [2 - (-2)]^2}\\ &= \sqrt{6^2 + 4^2}\\ &= \sqrt{36 + 16}\\ &= \sqrt{52}\end{aligned}$$

The length of the median from vertex P is $\sqrt{52}$.

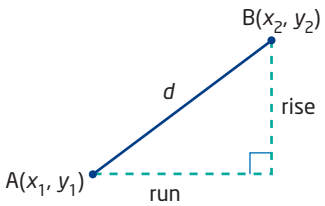
Key Concepts

- You can calculate the length, d, of a line segment using its run and rise:

  $d = \sqrt{(\text{run})^2 + (\text{rise})^2}$

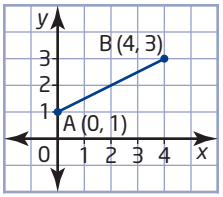
- You can also determine the length of a line segment directly from the coordinates of its endpoints, (x_1, y_1) and (x_2, y_2):

  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Communicate Your Understanding

C1 Describe how you can find the length of the line segment joining the points A(0, 1) and B(4, 3).



C2 When you use the formula for the length of a line segment, does it matter which point is represented by (x_1, y_1) and which point is represented by (x_2, y_2)? Use an example to explain your reasoning.

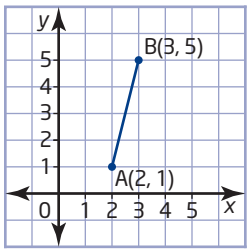
C3 Explain why the expression $(x_2 - x_1)^2 + (y_2 - y_1)^2$ never has a negative value.

Practise

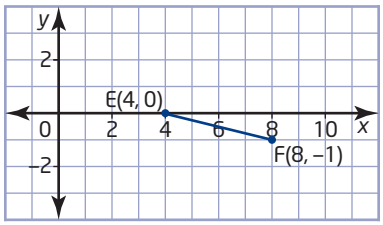
For help with questions 1 to 3, see Examples 1 to 3.

1. Estimate the length of each line segment from its graph. Then, calculate the exact length.

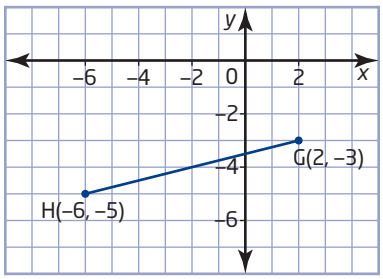
a)



b)



c)



2. Calculate the length of the line segment defined by each pair of endpoints.

a) A(−6, −2) and B(4, 3)

b) C(−2, 0) and D(7, −3)

c) E(−5, −6) and F(−1, −2)

d) G(0, 5) and H(8, −1)

3. Calculate the length of the line segment defined by each pair of endpoints.

a) J(2.1, 8.3) and K(−4.5, −4.7)

b) L(−4.2, −5.1) and M(11.6, 9.2)

c) $N\left(\frac{1}{2}, \frac{5}{2}\right)$ and $P\left(\frac{3}{2}, -\frac{5}{2}\right)$

Connect and Apply

4. On a city map, the coordinates of two department stores are (4, 3) and (1, 7). How far apart are the stores if each unit on the map represents 1 km?

5. On a street map of his town, Jordan's house has coordinates (8, 1). The town's two high schools are at (0, 5) and (6, 11).

a) Which school is closer to Jordan's house?

b) Describe a method you could use to check your answer to part a).

6. The vertices of △ABC are A(2, 5), B(−6, −1), and C(10, −1).

a) Determine the length of each side of this triangle.

b) Determine the perimeter of the triangle.

c) Classify the triangle.

7. a) Show that the triangle with vertices D(−1, 0), E(1, 0), and F(0, $\sqrt{3}$) is equilateral.

b) List the coordinates of the vertices of another equilateral triangle.

8. Determine the length of the median from vertex J in the triangle with vertices J(−2, −2), K(−3, 2), and L(1, 3).

9. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 8.

10. Determine the area of the right triangle with vertices R(4, 4), S(−2, −2), and T(10, −2).

11. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 10.

12. Use the length formula to verify that C(−5, −1) is the midpoint of the line segment joining A(−2, 5) and B(−8, −7).

13. A line segment has endpoints K(−2, 7) and L(4, −2).

a) Find the coordinates of the midpoint of this line segment.

b) Use the length formula to verify your answer to part a).

14. An architect's drawing shows a pipe running diagonally under a basement floor from a floor drain to a sewer connection. The floor drain is at a point 2 m east and 2 m north of the southwest corner of the basement. The sewer connection is 10 m east and 17 m north of the corner. The pipe costs $3.15 per metre, including taxes. How much should the builder budget for pipe for the floor drain?

15. **a)** Draw a triangle with vertices P(−3, −4), Q(5, 1), and R(2, 7).

b) Determine the coordinates of the midpoints of PQ and PR. Label these midpoints S and T.

c) Show that ST is half the length of QR.

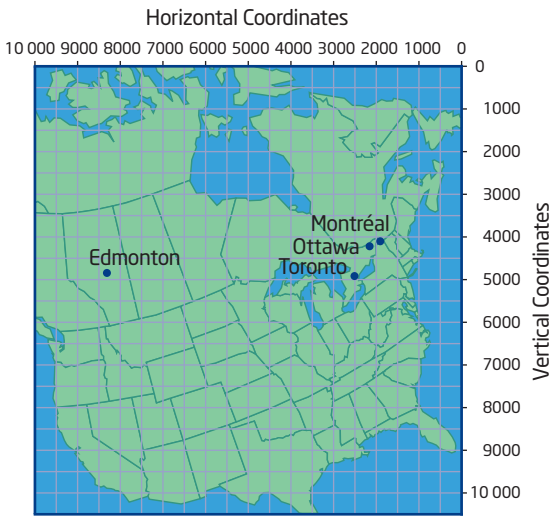
d) Show that ST is parallel to QR.

e) Show that the triangle formed by joining the midpoints of the sides of △PQR is similar to △PQR.

16. **Use Technology** Use *The Geometer's Sketchpad*® or Cabri® Jr. to verify your answer to question 15.

17. The charges for most long-distance telephone calls used to be based on the distance between the two stations and the duration of the call. To determine the distance, telephone companies used a rectangular coordinate grid with its origin located off the northeast coast of Canada. The coordinates in this system indicate the horizontal and vertical distance from the grid's origin. This table lists the telephone coordinates, converted to kilometres, for four cities.

City	Coordinates
Edmonton	(3978, 2520)
Montréal	(1015, 2104)
Ottawa	(1142, 2232)
Toronto	(1268, 2540)



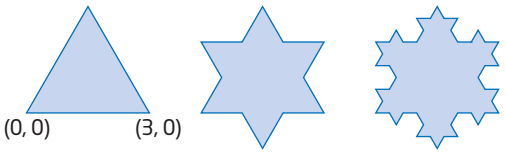
a) Calculate the distance, to the nearest kilometre, between Edmonton and Ottawa, between Montréal and Toronto, and between Edmonton and Toronto.

b) Research the flying distances between these cities. How accurate is the telephone coordinate system?

18. Chapter Problem Here are the first three stages of a Koch snowflake, named after the Swedish mathematician Niels Fabian Helge von Koch (1870–1924). This snowflake starts with an equilateral triangle. At each stage, the middle third of each side is replaced by two segments, each equal in length to the segment they replace.

a) Find the coordinates of the three new vertices on the bottom of the snowflake in the second stage.

b) Are Koch snowflakes fractals? Explain.



Go to www.mcgrawhill.ca/links/principles10 and follow the links to learn more about Koch snowflakes.

Achievement Check

19. Lightning starts a forest fire at a point with map coordinates F(23, 25). The nearest towns have coordinates A(15, 19) and B(23, 17).

a) Which town is at greater risk from the fire?

b) Describe any assumptions you made for your answer to part a).

c) At a campground midway between the two towns, a camper learns about the fire from a newscast. The camper estimates that he is about 9 km from the fire. Is the camper correct? Justify your response.

Extend

20. The point A(x, 1) is 5 units from the point (2, 6).

a) Find a possible value for x.

b) Is this value the only solution? Explain.

21. a) On a grid, draw line segments with the lengths listed below. Describe your method. How do you know that each line segment is the correct length?

i) $\sqrt{2}$ **ii)** $\sqrt{5}$ **iii)** $\sqrt{13}$ **iv)** $\sqrt{41}$

b) Compare the line segments you drew to those drawn by a classmate. Verify that both sets of line segments have the correct lengths.

22. List four points that satisfy each condition and have integer coordinates.

a) 5 units from the origin

b) 5 units from the point (2, 1)

c) 10 units from the point (−5, −2)

23. Sally has hidden her brother's birthday present somewhere in the backyard. When writing instructions for finding the present, she used a coordinate system with each unit on the grid representing 1 m. The positive y-axis of this grid points north. The instructions read "Start at the origin, walk halfway to (8, 6), turn 90° left, and then walk twice as far." Where is the present? How far is it from the origin?

24. Math Contest The number of arrangements of five letters from the word *magnetic* that contain the word *net* is

A 60 **B** 100 **C** 360 **D** 630 **E** 720

25. Math Contest Show that the area of the larger semicircle is equal to the sum of the areas of the two smaller semicircles.