2.4 Equation for a Circle

A licence is not required for portable two-way radios with a power of up to 2 W operating on the General Mobile Radio Service (GMRS) frequencies in Canada. GMRS radios are similar to Family Radio Service (FRS) radios, which are limited to 0.5 W and use different frequencies. Some radios are hybrids that can operate on both the GMRS and the FRS frequencies.

Investigate

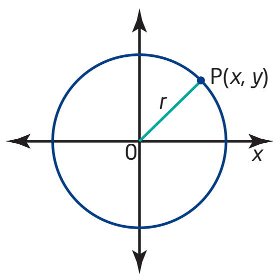
How can you find an equation for a circle?

Tools
- grid paper
- compasses

Near his home, Trevor's GMRS radios have a range of about 5 km.

Method 1: Use Pencil and Paper

1. Draw a circle to represent the range of Trevor's radios. Let the origin represent Trevor's position.
2. Label the x- and y-intercepts of your circle. What do these intercepts have in common?
3. Find four other points on the circle that have integer coordinates. Label these points A, B, C, and D, and mark their coordinates on your drawing. Use the distance formula to verify that each of these points is exactly 5 units from the origin.
4. Mark a point P(x, y) anywhere on the circle. Construct a right triangle with OP as the hypotenuse and the rise and the run of OP as the other two sides.



5. Write an equation relating the length of OP to the length of the other two sides of the right triangle. Substitute OP = 5 into the equation.

6. Verify that the coordinates of points A, B, C, and D satisfy the equation in step 5.

7. **Reflect** Will the coordinates of every other point on the circle also satisfy the equation? Explain your reasoning.

8. Away from built-up areas, Trevor finds that his GMRS radios have a range of about 7 km. Add a circle to your drawing to represent this larger range.

9. Write an equation for the larger circle.

10. **Reflect** Write an equation for the circle with centre (0, 0) and radius *r*. Then, use this equation to write an expression for the radius.

Tools

- computer with *The Geometer's Sketchpad®*

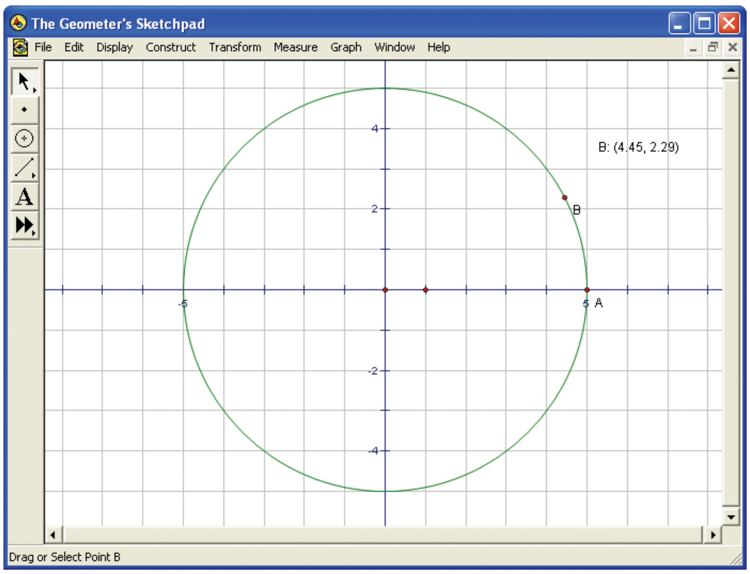
Method 2: Use *The Geometer's Sketchpad®*

1. Choose **Show Grid** from the **Graph** menu.

2. Construct a circle to represent the range of Trevor's radios. Let the origin represent Trevor's position. Use the **Compass Tool** to construct a circle with its centre at the origin and a radius of 5 units.

3. Label the *x*- and *y*-intercepts of the circle. What do these points have in common?

4. Use the **Point Tool** to construct a point on the circle. Select the point and choose **Coordinates** from the **Measure** menu. Then, drag the point around the circumference of the circle to find four other points that have integer coordinates. Construct these points and label them with their coordinates.



5. Verify that each of the points is 5 units from the origin. Select a point and the origin, and choose **Distance** from the **Measure** menu. Use the same method to measure the distance from the origin to each of the other points.

6. Construct a point anywhere on the circle. Label the point P. Construct a right triangle with OP as the hypotenuse and the rise and the run of OP as the other two sides.

7. Write an equation relating the length of OP to the length of the other two sides of the right triangle. Substitute OP = 5 into the equation.

8. Verify that the coordinates of the points in step 4 satisfy the equation in step 7.

9. **Reflect** Will the coordinates of every other point on the circle also satisfy the equation? Explain your reasoning.

10. Away from built-up areas, Trevor finds that his GMRS radios have a range of about 7 km. Add a circle to your drawing to represent this larger range.

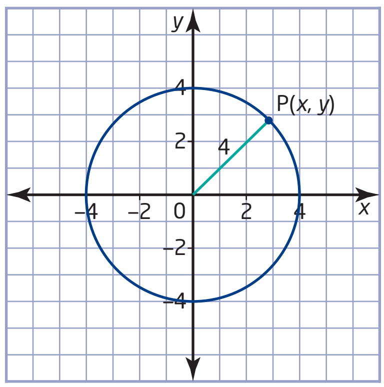
11. Write an equation for the larger circle.

12. **Reflect** Write an equation for the circle with centre (0, 0) and radius *r*. Then, use this equation to write an expression for the radius.

Example 1 Equation for a Circle

Find an equation for the circle with centre (0, 0) and radius 4.

Solution



The distance from the origin to any point P(x, y) on the circle is the length of the radius. So,

OP = 4

The distance formula also gives an expression for the length of OP:

$$\begin{aligned} OP &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(x - 0)^2 + (y - 0)^2} \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

Therefore,

$$\sqrt{x^2 + y^2} = 4$$

$$x^2 + y^2 = 16$$

An equation for the circle is $x^2 + y^2 = 16$.

Example 2 Determine Whether a Point Lies Within a Circle

a) Determine an equation and the radius for the circle that has its centre at the origin and passes through the point A(6, −8).

b) Is the point B(−5, 9) inside this circle?

Solution

a) An equation for a circle centred at the origin has the form $x^2 + y^2 = r^2$.

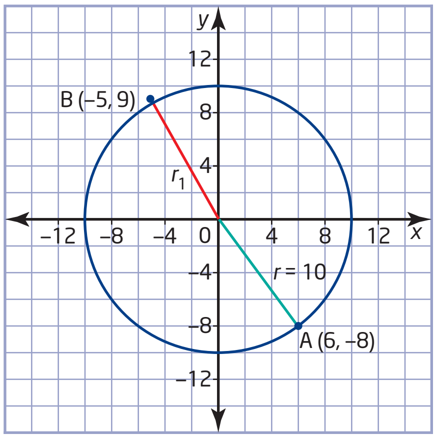
Substitute the coordinates of the point (6, −8) into the equation for the circle.

The point (6, −8) lies on this circle, so the coordinates of the point must satisfy the equation of the circle.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ 6^2 + (-8)^2 &= r^2 \\ 36 + 64 &= r^2 \\ 100 &= r^2 \\ \sqrt{100} &= \sqrt{r^2} \\ 10 &= r \end{aligned}$$

An equation for the circle is $x^2 + y^2 = 100$, and the radius of the circle is 10.

b) Consider a circle with its centre at the origin and with point B(−5, 9) on the circumference. Let r_1 be the radius of this circle. To find the length of the radius, substitute the coordinates of point B into the formula for the radius of a circle centred at the origin.



$$\begin{aligned} r_1 &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-5)^2 + 9^2} \\ &= \sqrt{25 + 81} \\ &= \sqrt{106} \\ &\doteq 10.3 \end{aligned}$$

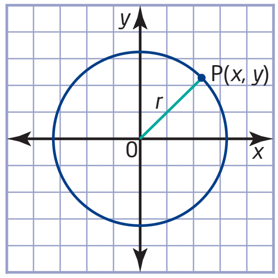
Since $r_1 > 10$, point B lies outside the circle defined by $x^2 + y^2 = 100$.

If $r_1 > r$, then $r_1^2 > r^2$. So, the inequality $x^2 + y^2 > r^2$ defines the region *outside* the circle with centre (0, 0) and radius r.

Reflect

Key Concepts

- An equation for the circle with centre at the origin and radius r is $x^2 + y^2 = r^2$.
- The radius of a circle centred at the origin is $r = \sqrt{x^2 + y^2}$.



Communicate Your Understanding

C1 Outline how you would find an equation for the circle centred at the origin with a radius of 12.

C2 Describe how you would determine whether the point (3, 5) lies on the circle defined by $x^2 + y^2 = 35$.

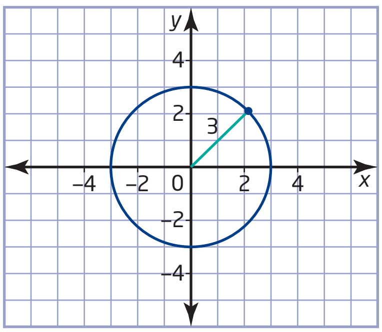
C3 Explain how you would determine whether the point (8, 8) lies inside the circle defined by $x^2 + y^2 = 100$.

Practise

For help with question 1, see Example 1.

1. Determine an equation for each circle.

a)



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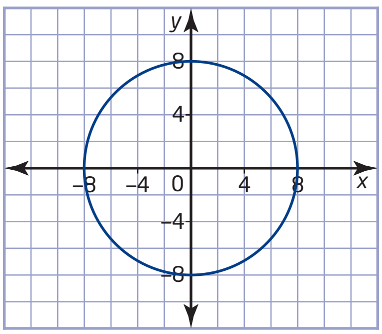

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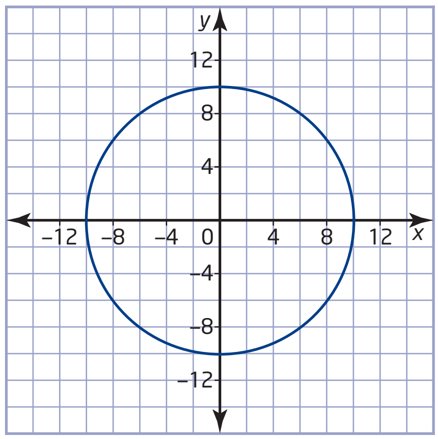

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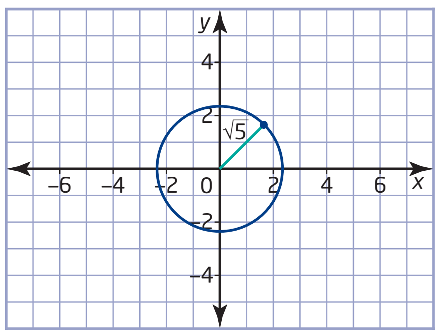

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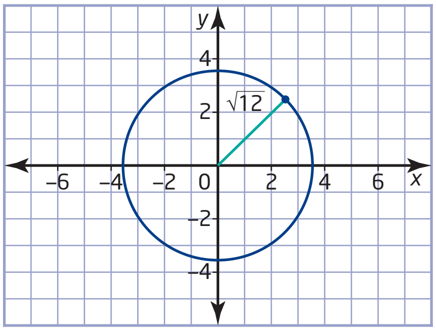

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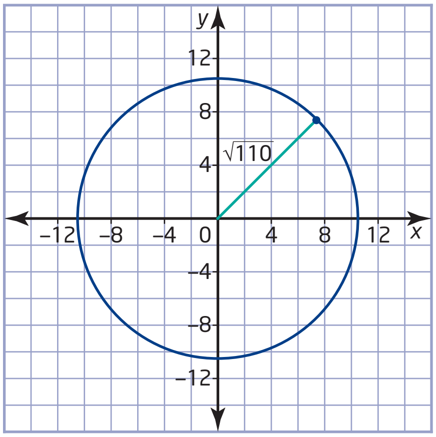


For help with questions 2 to 4, see Example 2.

2. For each equation, state the radius of the corresponding circle and give the coordinates of one point on the circle.

a) $x^2 + y^2 = 36$ **b)** $x^2 + y^2 = 144$

c) $x^2 + y^2 = 20$ **d)** $x^2 + y^2 = 50$

e) $x^2 + y^2 = 1.69$

3. For each point, find an equation for the circle that is centred at the origin and passes through the point. Then, check your answer by graphing the circle and plotting the point.

a) $(-4, 3)$ **b)** $(5, 2)$

c) $(-3, -6)$ **d)** $(-7, 12)$

4. Determine whether each point is on, inside, or outside the circle defined by $x^2 + y^2 = 34$.

a) $(5, -3)$ **b)** $(4, 4)$

c) $(-6, 0)$ **d)** $(-3, -5)$

e) $(2, -6)$ **f)** $(\sqrt{34}, 0)$

Connect and Apply

5. A satellite orbits Earth on a circular path with equation $x^2 + y^2 = 1.44 \times 10^8$, with distances measured in kilometres. Another satellite orbiting in the same plane passes through the point (8000, 9800). Is this satellite inside the orbit of the first one?

6. Determine an equation for the circle that has a diameter with endpoints $A(-4, 3)$ and $B(4, -3)$.

7. The point $(a, 8)$ lies on the circle defined by $x^2 + y^2 = 100$.

a) Explain why there are two possible values for a. Find these values.

b) Use a graph to check that the points corresponding to both values for a are on the circle.

8. A farmer is building a circular corral to hold livestock. With distances measured in metres, the shape of the corral is modelled by the equation $x^2 + y^2 = 64$.

a) Find the length of fencing required for this corral.

b) Find the area of the corral.

9. a) Graph the circle defined by $x^2 + y^2 = 100$.

b) Verify algebraically that the points $P(-8, 6)$ and $Q(6, 8)$ are both on the circle.

c) Find an equation for the right bisector of the **chord** PQ.

d) Verify that the right bisector in part c) passes through the centre of the circle.

e) Do you think that the right bisector of any chord of the circle passes through the centre of the circle? Explain your reasoning.

10. a) Graph the circle defined by $x^2 + y^2 = 40$.
b) Verify algebraically that the line segment joining R(−6, 2) and S(2, −6) is a chord of this circle.
c) Determine an equation for the line joining the centre O to the midpoint of this chord.
d) Verify that this line is perpendicular to the chord.

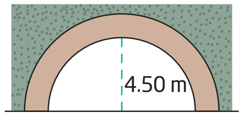
11. a) Graph the circle defined by $x^2 + y^2 = 41$.
b) Verify algebraically that the line segment joining U(−4, 5) and V(−5, −4) is a chord of this circle.
c) Determine an equation for the line that passes through the origin and is perpendicular to the chord UV.
d) Verify that this line passes through the midpoint of the chord.

12. **Use Technology** Use geometry software to determine whether the right bisector of any chord of a circle passes through the centre of the circle. Describe the method you used and your findings.

13. a) Graph the circle defined by $x^2 + y^2 = 25$.
b) Verify algebraically that the point A(−3, −4) lies on the circle.
c) Construct the line segment AO.
d) Draw the line through A that is perpendicular to AO. This perpendicular line is a **tangent of a circle**.
e) Determine an equation for the tangent in part d).
f) Explain why A is the only point that is on both the circle and the tangent.

14. Brandon has three close friends who live in different parts of the city. Brandon wants to meet them for lunch at a restaurant that is roughly equidistant from their homes. How could Brandon use his knowledge of circles to help find a suitable restaurant? Explain your reasoning.

15. As part of the North American Free Trade Agreement (NAFTA), Canada, the United States, and Mexico are developing joint standards for highway trucks. One standard specifies a maximum width of 2.60 m and a maximum height of 4.15 m. Will a truck of this size fit through a semicircular tunnel with a maximum height of 4.50 m?



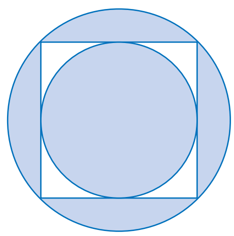
16. Lei is designing a construction set for small children. The set includes cylindrical cups with radii of 5 cm, 6 cm, and 7 cm as well as rectangular blocks that measure 7 cm by 8 cm by 9 cm. Will the blocks fit inside all of the cups?

17. A ship drops its anchor into the water and creates a circular ripple. The radius of this ripple increases at a rate of 50 cm/s.
a) Find an equation for the circle 10 s after the anchor is dropped.
b) A small rowboat is 50 m east and 75 m north of the point where the anchor was dropped. How long does the ripple take to reach the rowboat?
c) Describe any assumptions you made for your answers to parts a) and b).

Extend

18. Describe the region defined by each inequality. Then, draw and label a diagram of the three regions.
a) $x^2 + y^2 < 25$
b) $x^2 + y^2 > 49$
c) $25 < x^2 + y^2 < 49$

19. An equation for the small circle in this design is $x^2 + y^2 = 4$. Determine an equation for the larger circle.



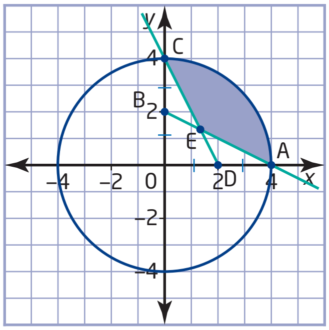
20. a) Write an equation for the circle in the diagram below.

b) Write the coordinates of points B and D.

c) Determine equations for the lines containing line segments AB and CD.

d) Determine the coordinates of point E.

e) Determine the area of the shaded portion of the circle.



21. A boat moving east at 2 m/s creates circular waves that travel outward from the boat at a speed of 1 m/s.

a) How far does the wave produced by the boat travel in the time it takes the boat to move 10 m?

b) Plot the waves when the boat has moved 10 m. Draw circles at 1-m intervals from the origin to (8, 0).

c) Describe the pattern formed by the points of intersection of the waves. What does this pattern represent?

22. Find an equation for the circle centred at (4, 3) with a radius of 5.

23. Use a counterexample to show that this statement is false: "Every circle with a radius greater than 1 has at least one point with integer coordinates."

24. a) Draw the triangle with vertices Q(−2, 0), R(2, 8), and S(7, 3). Then, construct the right bisector of each side.

b) Verify algebraically that the three right bisectors intersect at a single point, the **circumcentre** of △QRS.

c) Find the distance from each vertex of △QRS to the circumcentre.

d) Describe the circle that passes through the vertices of △QRS.

e) Describe how you would use geometry software to answer parts a) to d).

25. Math Contest Find the radius of the circle represented by the equation $kx^2 + ky^2 = r^2$, where $k > 0$.

26. Math Contest Given that $a > 0$ and $b > 0$, describe the graph of the equation $ax^2 + by^2 = r^2$ if

a) $a < b$

b) $a > b$

Making Connections

You can use *The Geometer's Sketchpad*® to generate a variety of fractals. One of the easiest is a nest of circles.

- From the **Edit** menu, choose **Preferences**. Click on the **Text** tab. Ensure that **For All New Points** is checked.
- Construct a horizontal line segment AB. Select AB and choose **Midpoint** from the **Construct** menu. Construct a circle with centre C and radius CB. Hide line segment AB.
- Select points A and B, in that order. Choose **Iterate** from the **Transform** menu. Map A onto A and B onto C. Choose **Add New Map** from the **Structure** menu. Map A onto C and B onto B.
- Press the + key to add iterations and the − key to remove iterations. When you are satisfied with the image, click on **Iterate**.
- Clean up the sketch by hiding all visible points.