3.4 Verify Properties of Quadrilaterals

The new addition to the Royal Ontario Museum features a number of quadrilateral panels. This controversial design mimics the shape of mineral crystals.

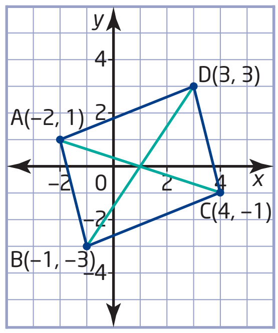
Investigate

Tools

- grid paper

How can you verify properties of a parallelogram?

Method 1: Use Pencil and Paper



1. Calculate the slopes of the sides of quadrilateral ABCD. Explain how you can use these slopes to verify that ABCD is a parallelogram.

2. Calculate the length of each side of ABCD. Explain how you can use these lengths to show that ABCD is a parallelogram.

3. Explain how you can use angle measurements to show that ABCD is a parallelogram.

4. **Reflect** List the properties that you can use to determine whether a given quadrilateral is a parallelogram.

5. Verify that the diagonals, AC and BD, bisect each other.

6. Explain how you can use congruent triangles to verify that the diagonals of ABCD bisect each other.

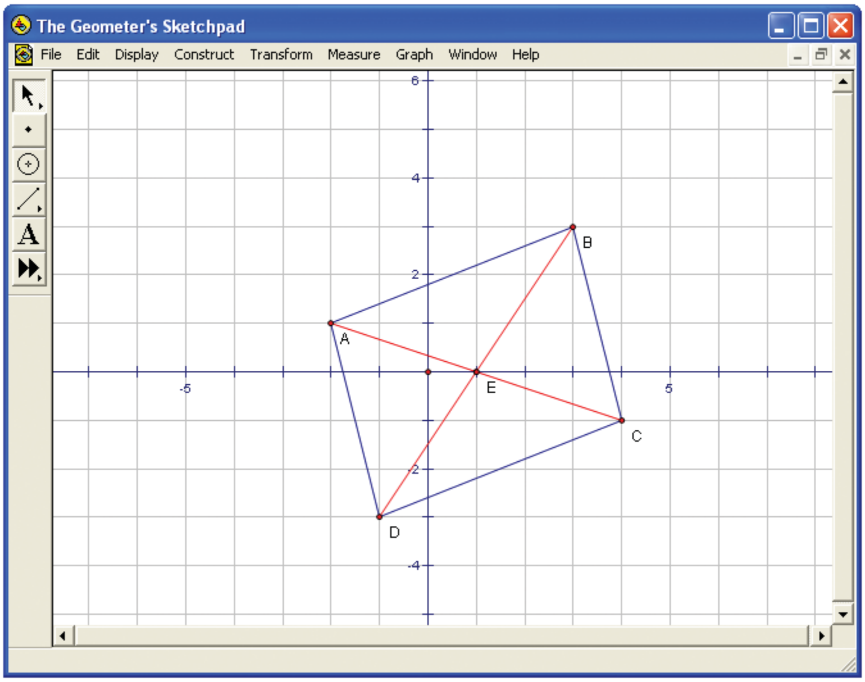
7. **Reflect** Are all quadrilaterals with diagonals that bisect each other parallelograms? Can you use this property to determine whether a given quadrilateral is a parallelogram? Justify your answer.

Tools

- computer with *The Geometer's Sketchpad®*

Method 2: Use *The Geometer's Sketchpad®*

1. Construct the quadrilateral with vertices A(−2, 1), B(3, 3), C(4, −1), and D(−1, −3).



2. Measure the slope of each side of the quadrilateral. Explain how you can use these slopes to verify that ABCD is a parallelogram.

3. Measure the length of each side of ABCD. Explain how you can use these lengths to show that ABCD is a parallelogram.

4. Explain how you can use angle measurements to show that ABCD is a parallelogram.

5. **Reflect** List the properties that you can use to determine whether a given quadrilateral is a parallelogram.

6. Construct the diagonals AC and BD and their point of intersection. Verify that the diagonals bisect each other.

7. **Reflect** Are all quadrilaterals with diagonals that bisect each other parallelograms? Can you use this property to determine whether a given quadrilateral is a parallelogram? Justify your answer.

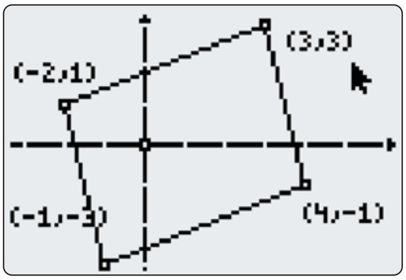
Method 3: Use a Graphing Calculator

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

1. Start the Cabri® Jr. application. Check that the axes are displayed.

2. Construct the quadrilateral with vertices A(−2, 1), B(3, 3), C(4, −1), and D(−1, −3). Choose **Coord. & Eq.** from the **F5** menu. Then, select the vertices to display their coordinates. Reposition any vertices that are not at the right coordinates.



3. Choose **Measure** from the **F5** menu, and **Slope** from the submenu. Measure the slope of each side of the quadrilateral. Explain how you can use these slopes to verify that ABCD is a parallelogram.

4. Measure the length of each side of ABCD. Explain how you can use these lengths to show that ABCD is a parallelogram.

5. Explain how you can use angle measurements to show that ABCD is a parallelogram.

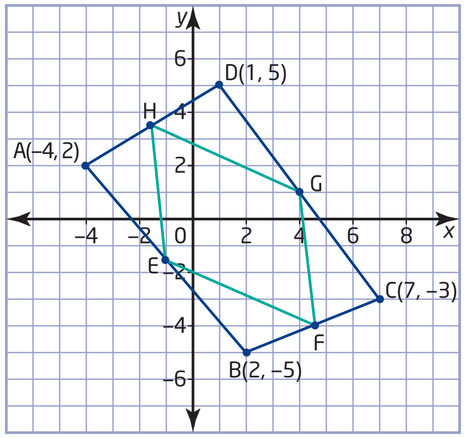
6. Reflect List the properties that you can use to determine whether a given quadrilateral is a parallelogram.

7. Construct the diagonals AC and BD and their point of intersection. Verify that the diagonals bisect each other.

8. Reflect Are all quadrilaterals with diagonals that bisect each other parallelograms? Can you use this property to determine whether a given quadrilateral is a parallelogram? Justify your answer.

Example 1 Midpoints of a Quadrilateral

Verify that quadrilateral EFGH, formed by joining the midpoints of adjacent sides of quadrilateral ABCD, is a parallelogram.



Solution

The simplest way to verify that EFGH is a parallelogram is to show that the slopes of the opposite sides are equal. First, find the coordinates of the midpoint of each side of ABCD.

$$\begin{aligned} E(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-4 + 2}{2}, \frac{2 + (-5)}{2}\right) \\ &= (-1, -1.5) \end{aligned}$$

$$\begin{aligned} F(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{2 + 7}{2}, \frac{-5 + (-3)}{2}\right) \\ &= (4.5, -4) \end{aligned}$$

$$\begin{aligned} G(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{7 + 1}{2}, \frac{-3 + 5}{2}\right) \\ &= (4, 1) \end{aligned}$$

$$\begin{aligned} H(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{1 + (-4)}{2}, \frac{5 + 2}{2}\right) \\ &= (-1.5, 3.5) \end{aligned}$$

Use these coordinates to find the slope of each side of EFGH.

$$\begin{aligned} m_{EF} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4 - (-1.5)}{4.5 - (-1)} \\ &= \frac{-2.5}{5.5} \times \frac{2}{2} \\ &= -\frac{5}{11} \end{aligned}$$

$$\begin{aligned} m_{FG} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-4)}{4 - 4.5} \\ &= \frac{5}{-0.5} \\ &= -10 \end{aligned}$$

$$\begin{aligned} m_{GH} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3.5 - 1}{-1.5 - 4} \\ &= \frac{2.5}{-5.5} \times \frac{2}{2} \\ &= -\frac{5}{11} \end{aligned}$$

$$\begin{aligned} m_{HE} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1.5 - 3.5}{-1 - (-1.5)} \\ &= \frac{-5}{0.5} \\ &= -10 \end{aligned}$$

Sides EF and GH have the same slope, so they are parallel. Similarly, FG is parallel to EH. Therefore, quadrilateral EFGH is a parallelogram.

Example 2 Properties of a Rhombus

a) Verify that the quadrilateral with vertices P(3, 3), Q(0, 1), R(3, −1), and S(6, 1) is a rhombus.

b) Verify that the diagonals of PQRS bisect each other at right angles.

Solution

a) Find the length of each side of PQRS.

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(0 - 3)^2 + (1 - 3)^2} \\ &= \sqrt{(-3)^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - 0)^2 + (-1 - 1)^2} \\ &= \sqrt{3^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} RS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6 - 3)^2 + (1 - (-1))^2} \\ &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} PS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6 - 3)^2 + (1 - 3)^2} \\ &= \sqrt{3^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

All four sides are equal in length. Therefore, PQRS is a rhombus.

b) If the diagonals have the same midpoint, they bisect each other.

Find the coordinates of the midpoint of each diagonal.

For PR:

$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{3 + 3}{2}, \frac{3 + (-1)}{2}\right) \\ &= (3, 1) \end{aligned}$$

For QS:

$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{0 + 6}{2}, \frac{1 + 1}{2}\right) \\ &= (3, 1) \end{aligned}$$

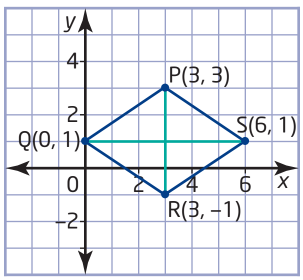
I could show that the diagonals bisect each other by calculating and comparing the lengths PT, QT, RT, and ST.

Since the midpoints of the diagonals have the same coordinates, the diagonals bisect each other.

Now, calculate the slopes of the diagonals.

$$\begin{aligned} m_{PR} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 - 3}{3 - 3} \\ &= \frac{-4}{0} \end{aligned}$$

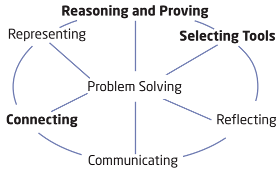
$$\begin{aligned} m_{QS} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - 1}{6 - 0} \\ &= 0 \end{aligned}$$



The graph of PQRS confirms that PR is vertical and QS is horizontal.

Since m_{PR} is undefined, PR is a vertical line segment. The zero value for m_{PR} indicates that QS is a horizontal line segment. So, PR and QS are perpendicular.

Therefore, PR and QS bisect each other at right angles.

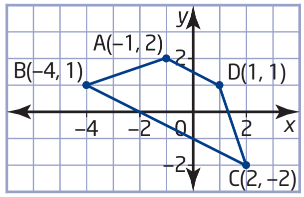


Key Concepts

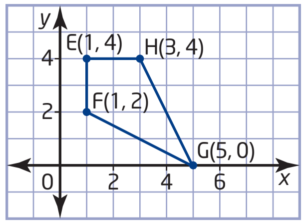
- You can use the formulas for lengths, midpoints, and slopes to verify properties of quadrilaterals.
- Often, there is more than one way to verify a property of a geometric shape.

Communicate Your Understanding

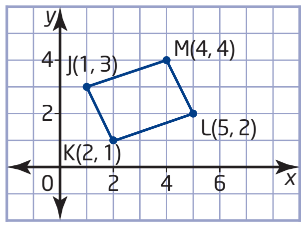
C1 Describe how to use analytic geometry to verify that quadrilateral ABCD is a trapezoid.



C2 Describe how to verify that the point of intersection of the diagonals of kite EFGH bisects only one of the diagonals.



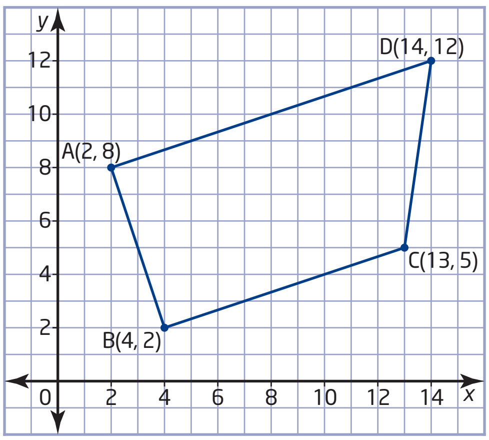
C3 Describe two methods for verifying that quadrilateral JKLM is a parallelogram. Which method is easier to use?



Practise

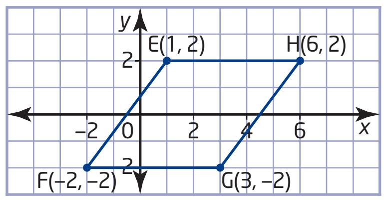
For help with question 1, see Example 1.

1. Verify that quadrilateral ABCD is a trapezoid.

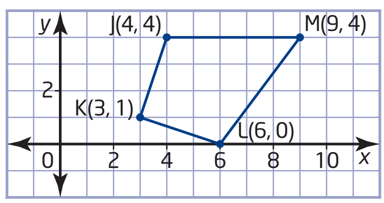


For help with questions 2 and 3, see Example 2.

2. Verify that quadrilateral EFGH is a rhombus.



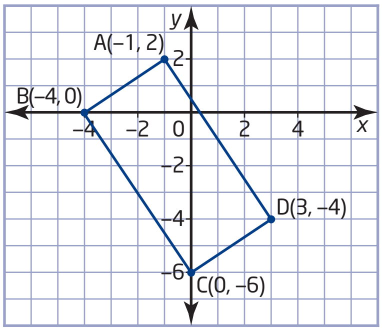
3. Verify that quadrilateral JKLM is a kite.



Connect and Apply

4. a) Verify that quadrilateral ABCD is a rectangle.

b) Verify that the diagonals of ABCD are equal in length and bisect each other.



5. a) Draw the quadrilateral with vertices P(0, 7), Q(−2, 1), R(4, −1), and S(6, 3).

b) Find the midpoint of each side. Join the midpoints of adjacent sides to form a new quadrilateral TUVW.

c) Verify that opposite sides of TUVW are parallel.

d) Verify that opposite sides of TUVW are equal in length.

6. Use Technology Use geometry software to answer question 5. Outline your method.

7. a) Draw the trapezoid with vertices A(−2, −2), B(2, −2), C(4, 1), and D(2, 4).

b) Verify that the line segment joining the midpoints of the non-parallel sides of the trapezoid is parallel to the other two sides.

8. Use Technology Use geometry software to answer question 7. Outline your method.

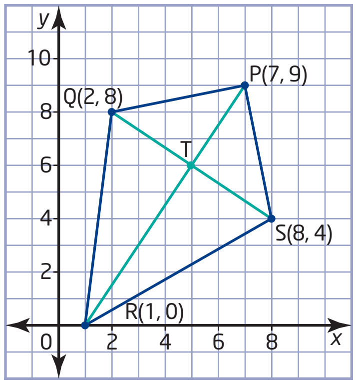
9. a) Verify that the diagonals of the rectangle with vertices J(−2, 1), K(2, 3), L(4, −1), and M(0, −3) bisect each other at right angles.

b) Do all rectangles have this property?

c) What can you conclude about the lengths of the sides of JKLM? Explain your reasoning.

10. a) Verify that PR bisects QS at right angles.

b) Verify that QS does not bisect PR.



11. a) Draw the quadrilateral with vertices A(3, 4), B(−1, 2), C(−3, −4), and D(5, −6). Then, join the midpoints of the adjacent sides of ABCD to form a new quadrilateral, EFGH.

b) Verify that EFGH is a rhombus.

c) Describe another method for verifying that EFGH is a rhombus.

12. a) Draw the quadrilateral with vertices P(−3, −1), Q(3, 1), R(7, 5), and S(1, 3). Then, draw the diagonals of PQRS.

b) Verify that the diagonals bisect each other.

c) What kind of quadrilateral is PQRS? Justify your answer.

13. Use Technology Use geometry software to answer question 12. Outline your method.

14. a) Draw the rhombus with vertices A(−5, 2), B(−1, 3), C(−2, −1), and D(−6, −2).

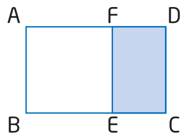
b) Verify that joining the midpoints of the adjacent sides of ABCD produces a rectangle.

15. Use Technology Use geometry software to answer question 14. Outline your method.

16. Chapter Problem

a) A rectangle with $\frac{\text{length}}{\text{width}} = \varphi$ is called a golden rectangle. On grid paper, draw a large rectangle, making the ratio of the length to the width as close to φ:1 as you can. The golden ratio, φ, equals 1.618….

b) Divide your golden rectangle into a square and a smaller rectangle, with the sides of the square equal to the width of the original rectangle. Measure the width and length of the smaller rectangle. Calculate the ratio of these dimensions.



c) Divide the smaller rectangle into a square and a third rectangle. Predict the ratio of the length and the width of the third rectangle. Measure these dimensions, and calculate their ratio. Do the measurements confirm your prediction?

d) Divide the third rectangle in the same way as the others to produce a fourth rectangle. Find the length-to-width ratio of this rectangle.

e) **Use Technology** Use *The Geometer's Sketchpad*® to construct a golden rectangle. Divide this rectangle into a square and a smaller rectangle. Divide the smaller rectangle in the same way. Continue the process, producing progressively smaller rectangles. Compare the length-to-width ratios of these rectangles. What can you conclude about these ratios?

f) Is the series of nested rectangles a fractal? Justify your answer.

g) Describe how you could use the nested rectangles to generate a golden spiral.

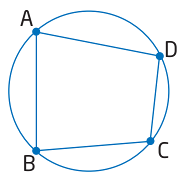
Extend

17. Let P(a, b), Q(c, d), R(e, f), and S(g, h) represent the coordinates of the vertices of a quadrilateral.

a) Determine the coordinates of the midpoints T, U, V, and W of sides PQ, QR, RS, and SP, respectively.

b) Verify that TUVW is a parallelogram.

18. Math Contest How are ∠BAD and ∠BCD related?



19. Math Contest How many four-digit numbers with no repeated digits can you make using the digits from 1 to 8?

A 7

B 8

C 1680

D 3584

E 4096